2. HARMONIC ANALYSIS ON COMPACT GROUPS.

These notes recall some general facts about Fourier analysis on a compact group K. They will be applied eventually to compact Lie groups, particularly to the maximal compact subgroups of real reductive Lie groups. But much of the early material makes no use of the Lie group structure, so I'll work without it for as long as possible.

Thanks to Ben Harris (2007) for correcting a number of slips of the keyboard and the mind.

You may wonder where in real life one might ever encounter a compact group that is *not* a Lie group. I know of two important places. First, suppose F is any field and $\overline{F}$ is an algebraic closure of F. The Galois group

$$\Gamma = F\text{-linear field automorphisms of } \overline{F}$$

has a natural compact topology. (It is the inverse limit of the Galois groups of the finite Galois extensions of F, and these are finite groups. The inverse limit topology makes Γ compact. A basic neighborhood of the identity consists of all automorphisms of $\overline{F}$ that are trivial on a specified finite Galois extension of F.)

For a second family of examples, the ring $\mathbb{Z}_p$ of p-adic integers is compact. Its additive group is therefore compact, as is the multiplicative group of invertible elements. More generally, the group $GL(n, \mathbb{Z}_p)$ (consisting of $n \times n$ matrices with entries in $\mathbb{Z}_p$ and determinant invertible in $\mathbb{Z}_p$) is compact.

In each of these example, the problem of parametrizing the set of irreducible representations is extremely complicated. For the second examples, part of the case $n = 2$ is treated in chapter III of [Silb]. To get some small hint about what's going on in the first example, try this. (Please recall that there is no homework; this is not to hand in!)

Exercise 2.1. *Suppose that F is a field of characteristic not two. Write Γ for the Galois group of $\overline{F}$ over F. Show that there is a bijection between the non-trivial one-dimensional representations of Γ taking values in $\{\pm 1\}$, and the quadratic extensions of F. If F is equal to $\mathbb{Q}$, deduce that such representations are in one-to-one correspondence with non-empty finite subsets of the "places" of $\mathbb{Q}$. (A "place" of $\mathbb{Q}$ is either a prime number or the symbol ∞.)*

Class field theory provides a complete parametrization of the one-dimensional representations of Galois groups of number fields (generalizing this exercise). Two-dimensional representations are much harder; although I am by no means an expert, I believe that there is no reasonable parametrization of them available.

We now turn to harmonic analysis. Always we work with a compact group K. A *finite-dimensional representation μ of K* consists of a finite-dimensional complex vector space V_μ, and a continuous group homomorphism

$$\mu: K \to \operatorname{Aut} V_\mu. \tag{2.1)(a}$$

In particular, this means that μ may be regarded as a continuous function on K with values in the finite-dimensional vector space $\operatorname{End} V_\mu$. If μ_1 and μ_2 are representations of K, the space of *intertwining operators from μ_1 to μ_2* is

$$\operatorname{Hom}_K(V_{\mu_1}, V_{\mu_2}) = \{T \in \operatorname{Hom}(V_{\mu_1}, V_{\mu_2}) \mid \mu_2(k)T = T\mu_1(k), \text{all } k \in K\}. \quad (2.1)(b)$$

We say that μ_1 and μ_2 are *equivalent* if there is an invertible intertwining operators from μ_1 to μ_2; this is an equivalence relation on representations of K.

An *invariant subspace* of μ is a subspace $W \subset V_\mu$ with the property that

$$\mu(k)W \subset W, \qquad \text{all } k \in K.$$

The subspaces V_μ and 0 are always invariant. We say that μ is *irreducible* if it has exactly two invariant subspaces; that is, if $V_\mu \neq 0$, and there are no non-trivial invariant subspaces. The *dual object of* K is

$$\widehat{K} = \{\text{equivalence classes of irreducible representations of } K\}. \qquad ((2.1)(c)$$

Every compact group has a *trivial representation* on the vector space $\mathbb{C}$, consisting of the trivial homomorphism from K to $\mathbb{C}^\times$. It is irreducible. (The representation of K on the zero vector space is *not* irreducible, because it has only one invariant subspace, rather than the required two.)

If K is the circle group

$$S^1 = \{e^{i\theta} \mid \theta \in \mathbb{R}\},$$

then all of the irreducible representations are one-dimensional. For every integer n, we have a representation μ_n on $\mathbb{C}$ defined by

$$\mu_n(e^{i\theta}) = e^{in\theta} \in \mathbb{C}^\times = \operatorname{Aut} \mathbb{C}. \qquad (2.2)(a)$$

These representations are irreducible and inequivalent, and every irreducible representation is equivalent to one of them. Therefore

$$\widehat{S^1} = \mathbb{Z}. \qquad (2.2)(b)$$

Harmonic analysis for K relates (things like) functions on K with (things like) functions on $\widehat{K}$. To get such relationships, we need a Fourier transform (denoted by a hat) carrying functions on K to functions on $\widehat{K}$; and an inverse Fourier transform (denoted by an inverted hat) going in the other direction. In the case of S^1, the usual Fourier transform carries a (complex-valued continuous) function f on S^1 to the function $\hat{f}$ on $\widehat{S^1} = \mathbb{Z}$ defined by

$$\hat{f}(n) = \frac{1}{2\pi}\int_0^{2\pi} f(e^{i\theta})e^{in\theta}d\theta. \qquad (2.2)(c)$$

(This is usually written $\hat{f}(-n)$, but the present normalization will be more convenient for us.) The first thing to notice about this formula is that it is more naturally written not for the function f, but for the measure $fd\theta$. The second thing to notice is that it makes sense for much more general measures: certainly for regular Borel measures, and even for distributions or hyperfunctions on the circle.

For the inverse Fourier transform, if a is any complex-valued function on $\mathbb{Z}$, we can try to define

$$\check{a}(e^{i\theta}) = \sum_{n\in\mathbb{Z}} a(n)e^{-in\theta}. \tag{2.2)(d}$$

(Again this differs from standard formulas by a minus sign.) In contrast to the situation with the Fourier transform, which easily extended to much larger domains, this definition does not make sense without some restrictive hypothesis on the function a. The easiest hypothesis is that a has finite support, but of course making the definition work under weaker hypotheses is a problem of tremendous interest. On the other hand, if a *does* have finite support, then its inverse Fourier transform is much better than continuous: it is smooth, and even analytic.

With this example as motivation, we can formulate some general definitions. Begin with

$$C(K) = \{\text{complex-valued continuous functions on } K\}.$$

This is a Banach space with norm given by the maximum of the absolute value. It's convenient to have also the space

$$\mathcal{M}(K) = \{\text{complex-valued regular Borel measures on } K\}.$$

You can read about such measures in many books about measure theory or functional analysis, and learn how to regard $\mathcal{M}(K)$ as a Banach space as well. (See for example [Rud], 6.18.) We will use the fact that if $f \in C(K)$ and $\delta \in \mathcal{M}(K)$, then

$$\langle f, \delta\rangle = \int_K f(k)\delta(k) \in \mathbb{C}$$

is well-defined (and finite). According to the Riesz Representation Theorem, this integration identifies $\mathcal{M}(K)$ as the dual Banach space to $C(K)$. (So far we need only the fact that K is a compact topological space.) If V is any finite-dimensional complex vector space, then

$$C(K;V) = \{V\text{-valued continuous functions on } K\} \simeq C(K) \otimes_{\mathbb{C}} V.$$

If $F \in C(K;V)$ and $\delta \in \mathcal{M}(K)$, then we can define

$$\langle F, \delta\rangle = \int_K F(k)\delta(k) \in V,$$

for example by choosing a basis of V and integrating each coordinate function.

Definition 2.3. Suppose K is a compact group, μ is a representation of K, and $\delta \in \mathcal{M}(K)$ is a regular Borel measure on K. The *operator-valued Fourier transform of δ at μ* is

$$\widehat{\delta}(\mu) = \langle \mu, \delta\rangle = \int_K \mu(k)\delta(k) \in \operatorname{End} V_\mu.$$

This is a reasonable generalization of the classical definition in (2.2)(c). In particular, the Fourier transform is a kind of function on the dual object $\widehat{K}$. It takes values not in the complex numbers, but in a separate algebra $\operatorname{End} V_\mu$ at each irreducible representation μ. (You should check that if μ and μ' are equivalent irreducible representations, then the two algebras $\operatorname{End} V_\mu$ and $\operatorname{End} V_{\mu'}$ are canonically isomorphic.)

The dual object for K is playing a role analogous to that of $\operatorname{Spec} A$ in commutative algebra. We relate the abstract commutative algebra A to an algebra of functions on $\operatorname{Spec} A$, by sending a to the function whose value at a prime ideal $\mathfrak{p}$ is the image of a in the field of fractions of $A/\mathfrak{p}$. We need to use functions taking different values at different points of $\operatorname{Spec} A$.

In the compact group setting one can form the algebra

$$\operatorname{Op}(\widehat{K}) = \prod_{\mu \in \widehat{K}} \operatorname{End} V_\mu, \tag{2.4}$$

which is where the operator-valued Fourier transform takes values. (The algebra structure is defined coordinate by coordinate.) For commutative algebras, the most important fact about the map from A to functions on $\operatorname{Spec} A$ is that it is an algebra homomorphism. To state the corresponding fact for compact groups, we need an algebra structure on $\mathcal{M}(K)$. That's provided by convolution of measures. To define it, we need two general constructions. If δ_1 and δ_2 are regular Borel measures on compact sets K_1 and K_2, then we can form their product $\delta_1 \boxtimes \delta_2$, which is a regular Borel measure on $K_1 \times K_2$. Its characteristic property is that if f_i is a continuous function on K_i, and we define $(f_1 \boxtimes f_2)(k_1, k_2) = f_1(k_1) f_2(k_2)$, then

$$\int_{K_1 \times K_2} (f_1 \boxtimes f_2)(\delta_1 \boxtimes \delta_2) = \left(\int_{K_1} f_1 \delta_1 \right) \left(\int_{K_2} f_2 \delta_2 \right).$$

Next, suppose $\phi\colon K_1 \to K_2$ is a continuous map. Composition with ϕ defines a continuous linear map of Banach spaces (*pullback of functions*)

$$\phi^*\colon C(K_2) \to C(K_1), \qquad \phi^*(f_2) = f_2 \circ \phi.$$

The transpose of this linear map is a continuous linear map of the dual Banach spaces (*pushforward of measures*)

$$\phi_*\colon \mathcal{M}(K_1) \to \mathcal{M}(K_2), \qquad \langle f_2, \phi_* \delta_1 \rangle = \langle \phi^* f_2, \delta_1 \rangle.$$

A little more explicitly,

$$\int_{K_2} f_2(k_2)(\phi_*(\delta_1)(k_2)) = \int_{K_1} (f_2 \circ \phi(k_1)) \delta_1(k_1).$$

Finally, suppose that δ_1 and δ_2 are regular Borel measures on K. Multiplication on K is a continuous map

$$m\colon K \times K \to K, \qquad m(k_1, k_2) = k_1 k_2.$$

The *convolution of* δ_1 *with* δ_2 is the regular Borel measure $\delta_1 * \delta_2$ on K defined by

$$\delta_1 * \delta_2 = m_*(\delta_1 \boxtimes \delta_2). \tag{2.5(a)}$$

A little more explicitly, if f is a continuous function on K,

$$\int_K f(k)(\delta_1 * \delta_2)(k) = \int_K \int_K f(k_1 k_2)\delta_1(k_1)\delta_2(k_2). \tag{2.5)(b}$$

It's easy to pass from here to parallel formulas for (say) $((\delta_1 * \delta_2) * \delta_3)$. From these formulas, the associativity of $\boxtimes$ on measures, and associativity of multiplication in K, it follows that the convolution product is an associative algebra structure on $\mathcal{M}(K)$.

The formula (2.5)(b) may look entirely unrelated to other formulas for convolution you may have seen, like the one for convolving two (appropriately integrable) functions on the real line:

$$(h_1 * h_2)(x) = \int_{\mathbb{R}} h_1(x-y)h_2(y)dy.$$

To bring them closer, recall that K has a *Haar measure* dk: this is characterized up to scalar multiplication as the positive regular Borel measure that is unchanged by left or right translation by elements of K. Using dk we can map functions to measures:

$$C(K) \to \mathcal{M}(K), \qquad h \mapsto h \cdot dk. \tag{2.5)(c}$$

Suppose now that h_1 and h_2 are continuous functions on K, and $\delta_i = h_i \cdot dk$ the corresponding measures. We compute

$$\int_K f(k)(\delta_1 * \delta_2)(k) = \int_K \int_K f(k_1 k_2)h_1(k_1)dk_1 h_2(k_2)dk_2.$$

Now change variables in the k_1 integration, replacing k_1 by $x = k_1 k_2$. This does not affect the measure dk_1, so we get

$$\int_K \int_K f(x)h_1(xk_2^{-1})dx h_2(k_2)dk_2.$$

Use Fubini's theorem to interchange the order of integration, to get

$$\int_K \int_K f(x)\left[h_1(xk_2^{-1})h_2(k_2)dk_2\right]dx.$$

The conclusion is that $\delta_1 * \delta_2 = h(k)dk$, with h the continuous function

$$h(x) = \int_K h_1(xy^{-1})h_2(y)dy. \tag{2.5)(d}$$

That is, the inclusion (2.5)(c) and the convolution product on measures define a convolution product on continuous functions, by the formula

$$(h_1 * h_2)(x) = \int_K h_1(xy^{-1})h_2(y)dy. \tag{2.5)(e}$$

This formula is closer to the one for $\mathbb{R}$ than (2.5)(b). For our present purposes it is less convenient and less natural, because of the need to choose a Haar measure.

Theorem 2.6. *Suppose K is a compact group. The operator-valued Fourier transform of Definition 2.3 is an algebra homomorphism from the regular Borel measures $\mathcal{M}(K)$ (with the convolution product of (2.5)) to $\mathrm{Op}(\widehat{K})$ (cf. (2.4)). The unit mass supported at the identity element of K maps to the identity element of $\mathrm{Op}(\widehat{K})$.*

The proof involves only understanding the definitions, and so I will leave it as a good exercise.

Before we continue, it is convenient to introduce various actions of K. Each element $k \in K$ defines left and right translation operators on continuous functions:

$$\lambda(k)\colon C(K) \to C(K), \qquad (\lambda(k)f)(x) = f(k^{-1}x), \tag{2.7)(a}$$

$$\rho(k)\colon C(K) \to C(K), \qquad (\rho(k)f)(x) = f(xk). \tag{2.7)(b}$$

The distribution of inverses ensures that

$$\lambda(k_1k_2) = \lambda(k_1)\lambda(k_2), \qquad \rho(k_1k_2) = \rho(k_1)\rho(k_2). \tag{2.7)(c}$$

In fact λ and ρ are continuous representations of K on $C(K)$, in the sense that maps like

$$K \times C(K) \to C(K), \qquad (k, f) \mapsto \lambda(k)f$$

are continuous.[1] Taking inverse transpose defines translation on measures:

$$\lambda(k)\colon \mathcal{M}(K) \to \mathcal{M}(K), \qquad \langle f, \lambda(k)\delta\rangle = \langle \lambda(k^{-1})f, \delta\rangle, \tag{2.7)(d}$$

and similarly for ρ. Again these actions respect multiplication in K, but they are *not* continuous representations unless K is finite: small translations of a point mass do not converge to the point mass in the Banach space topology on $\mathcal{M}(K)$. Finally, if μ_1 and μ_2 are two representations of K, we can define representations λ and ρ on $\mathrm{Hom}(V_{\mu_1}, V_{\mu_2})$ by

$$\lambda(k)T = \mu_2(k) \circ T, \qquad \rho(k)T = T \circ \mu_1(k^{-1}). \tag{2.7)(e}$$

These representations exist in particular on each endomorphism algebra $\mathrm{End}\, V_\mu$, so we get actions λ and μ of K on $\mathrm{Op}(\widehat{K})$. In each of these settings the left and right actions commute, so we can think of them as a single action (λ, ρ) of the group $K \times K$.

Proposition 2.8. *The operator-valued Fourier transform respects left and right translation:*

$$\widehat{\lambda(k)\delta} = \lambda(k)\widehat{\delta}, \qquad \widehat{\rho(k)\delta} = \rho(k)\widehat{\delta}.$$

Proof. We must show that if μ is any irreducible representation of K, δ is a measure on K, and $k \in K$, then

$$\langle \mu, \lambda(k)\delta\rangle = \mu(k)\langle \mu, \delta\rangle.$$

[1] There is a dangerous bend here. If you studied only Banach spaces among topological vector spaces, then you may think that the only reasonable topology on $\mathrm{End}(C(K))$ is the Banach space topology coming from the operator norm. With that topology, maps like $\lambda\colon K \to \mathrm{End}(C(K))$ are almost never continuous. In order to define continuous representations in terms of continuity of the map λ, one needs to use instead the strong operator topology on $\mathrm{End}(C(K))$.

The left side is by definition

$$\int_K \mu(x)(\lambda(k)\delta)(x) = \int_K (\lambda(k^{-1})\mu)(x)\delta(x) = \int_K \mu(kx)\delta(x) = \mu(k)\int_K \mu(x)\delta(x).$$

The last expression here is the right side, as we wished to show. The proof for ρ is identical. □

The next task is to find a way to go back: from operators on representations to functions on K. One standard way to extract functions from representations is using matrix coefficients. Suppose we are given a representation μ of K on a finite-dimensional *Hilbert* space V_μ, and v and w are vectors in V_μ. Then the matrix coefficient

$$f^\mu_{v,w}(k) = \langle \mu(k^{-1})v, w\rangle \tag{2.9)(a}$$

is a continuous function on K. (The inverse on the k is meant to simplify matters later on; many treatments omit it.) We want another way to think about this function, avoiding the choice of a Hilbert space structure on V_μ. Write V_μ^* for the dual vector space of V_μ. This space carries a representation μ^* of K, defined by

$$\mu^*(k) = \mu(k^{-1})^t. \tag{2.9)(b}$$

That is, if $\xi \in V_\mu^*$ and $v \in V_\mu$, and we write $(\cdot,\cdot)$ for the pairing between V_μ and V_μ^*,

$$(v, \mu^*(k)\xi) = (\mu(k^{-1})v, \xi). \tag{2.9)(c}$$

Given the Hilbert space structure, the vector w corresponds to a linear functional $\xi \in V_\mu^*$, by

$$(v', \xi) = \langle v', w\rangle. \tag{2.9)(d}$$

(The map from V_μ to V_μ^* sending w to ξ is a conjugate-linear isomorphism.) We can therefore think of a matrix coefficient as given by choosing $v \in V_\mu$, $\xi \in V_\mu^*$, and defining

$$f^\mu_{v,\xi}(k) = (\mu(k^{-1})v, \xi). \tag{2.9)(e}$$

It is very easy to compute the action of left and right translation on this function:

$$\lambda(k)f^\mu_{v,\xi} = f^\mu_{\mu(k)v,\xi}, \qquad \rho(k)f^\mu_{v,\xi} = f^\mu_{v,\mu^*(k)\xi}. \tag{2.9)(e}$$

Now the pair (v,ξ) also defines an endomorphism of V_μ:

$$T_{v,\xi} \in \operatorname{End} V_\mu, \qquad T_{v,\xi}(u) = (u,\xi)v. \tag{2.9)(f}$$

The trace of the endomorphism $T_{v,\xi}$ is (v,ξ). We can therefore write the matrix coefficient as

$$f^\mu_{v,\xi}(k) = \operatorname{tr}[\mu(k^{-1})T_{v,\xi}]. \tag{2.9)(g}$$

This formulation at last suggests a generalization we can use.

Definition 2.10. Suppose K is a compact group, μ is a representation of K, and $T \in \operatorname{End} V_\mu$. The *inverse Fourier transform of* T is the function

$$f_T(k) = \operatorname{tr}[\mu(k^{-1})T] = \operatorname{tr}[T\mu(k^{-1})] \in C(K).$$

Proposition 2.11. *Suppose K is a compact group, and μ is a representation of K. The range of the inverse Fourier transform*

$$\operatorname{End} V_\mu \to C(K), \qquad T \mapsto f_T$$

(Definition 2.10) is precisely the linear span of the set of matrix coefficients of μ (cf. (2.9)). This map respects the left and right actions of K:

$$\lambda(k) f_T = f_{\lambda(k)T}, \qquad \rho(k) f_T = f_{\rho(k)T}.$$

Proof. The first assertion is immediate from the discussion around (2.9). In light of the definitions (2.7)(a), (2.7)(e), and 2.10, the second assertion says that

$$f_T(k^{-1}x) = f_{\mu(k)T}(x),$$

or

$$\operatorname{tr}\left[\mu(x^{-1}k)T\right] = \operatorname{tr}\left[\mu(x^{-1})(\mu(k)T)\right].$$

This is clear from the fact that μ is a homomorphism. The assertion about ρ is identical. □

In order to state the basic facts about Fourier inversion on K, it is helpful to introduce some objects that get rid of the analytical problems. (In the case of the circle, what we are doing is concentrating on trigonometric polynomials.) The subscript F stands for "finite." Define

$$C(K)_F = \{f \in C(K) \mid \lambda(K)f \text{ spans a finite-dimensional space}\} \qquad (2.12)(a)$$

$$\mathcal{M}(K)_F = \{\delta \in \mathcal{M}(K) \mid \lambda(K)\delta \text{ spans a finite-dimensional space}\} \qquad (2.12)(b)$$

$$\begin{aligned} \operatorname{Op}(\widehat{K})_F &= \{T \in \operatorname{Op}(\widehat{K}) \mid \lambda(K)T \text{ spans a finite-dimensional space}\} \qquad (2.12)(c) \\ &= \sum_{\mu \in \widehat{K}} \operatorname{End} V_\mu. \end{aligned}$$

(The second equality in the last definition is a fairly easy exercise, using the fact that the representation of K on $\operatorname{End} V_\mu$ by λ is a sum of $\dim V_\mu$ copies of μ.) These objects will all be referred to informally as "K-finite." The space $C(K)_F$ of K-finite continuous functions is a subalgebra of $C(K)$: the reason is that the space of left translates of $f_1 f_2$ is the image under a linear map of the tensor product of the spaces of left translates of f_1 and f_2. (The linear map sends $f_1' \otimes f_2'$ to $f_1' f_2'$.) For similar reasons, the space of K-finite distributions is a subalgebra under convolution; this uses the easy (but not quite obvious) fact that

$$\lambda(k)(\delta_1 * \delta_2) = (\lambda(k)\delta_1) * (\lambda(k)\delta_2).$$

Finally, the second formula for the K-finite part of the operator algebra shows clearly that it is an algebra. The only slightly dangerous bend is that (for K infinite) this algebra has no identity element; what ought to be the identity (taking the identity operator in every $\operatorname{End} V_\mu$) is not in the algebraic direct sum. Similarly, the convolution algebra $\mathcal{M}(K)_F$ has no identity element if K is infinite, because the unit mass at the identity generates the infinite-dimensional space of all finite mass distributions under left translation.

Here is a version of the Peter-Weyl theorem for K.

Theorem 2.13. *Suppose K is a compact group. Fix a Haar measure dx on K, and write $\mathrm{Vol}(K)$ for the measure of K.*

1. *Multiplication by dx is a linear isomorphism from $C(K)_F$ onto $\mathcal{M}(K)_F$ (cf. (2.12)) respecting the representations λ and ρ of K.*
2. *The operator-valued Fourier transform (Definition 2.3) is an algebra isomorphism from $\mathcal{M}(K)_F$ onto $\mathrm{Op}(\widehat{K})_F$, respecting the representations λ and ρ of K.*
3. *The inverse Fourier transform (Definition 2.10) is a linear isomorphism of $\mathrm{Op}(\widehat{K})_F$ onto $C(K)_F$ respecting the representations λ and ρ of K.*
4. *For each $\mu \in \widehat{K}$, define*

$$d(\mu) = \frac{\mathrm{Vol}(K)}{\dim V_\mu}.$$

Then the composition of the three isomorphisms above acts as multiplication by $d(\mu)$ on $\mathrm{End}\, V_\mu$. That is, for any $T \in \mathrm{End}\, V_\mu$,

$$[\widehat{f_T(x)dx}](\mu') = \begin{cases} d(\mu)T & \text{if } \mu' = \mu \\ 0 & \text{if } \mu' \neq \mu. \end{cases}$$

Proof. Part (1) is immediate from the translation invariance of Haar measure. Part (2) (with "isomorphism onto" replaced by "homomorphism") is a consequence of Theorem 2.6 and Proposition 2.8. Part (3) (with "isomorphism onto" replaced by "map") is a consequence of Proposition 2.11. Consider part (4). The vector space $\mathrm{End}\, V_{\mu'}$ carries the irreducible representation $\mu' \boxtimes (\mu')^*$ of $K \times K$, by the actions λ and ρ. The map

$$\mathrm{End}\, V_\mu \to \mathrm{End}\, V_{\mu'}, \qquad T \mapsto [\widehat{f_T(x)dx}](\mu')$$

intertwines the actions λ and ρ (by what we have already proved); so by Schur's lemma, this map is zero if $\mu \neq \mu'$, and equal to some scalar $c(\mu)$ if $\mu = \mu'$. That is,

$$[\widehat{f_T(x)dx}](\mu) = c(\mu)T.$$

Explicitly, this means that

$$\int_K f_T(x)\mu(x)dx = c(\mu)T,$$

or

$$\int_K [\mathrm{tr}\,\mu(x^{-1})T]\mu(x)dx = c(\mu)T \qquad (T \in \mathrm{End}\, V_\mu). \tag{2.14)(a}$$

To prove (4), it remains to calculate $c(\mu)$. Fix a vector $w \in V_\mu$ and a linear functional $\tau \in V_\mu^*$. Apply the equation (2.14)(a) to the vector w, and then apply the linear functional τ. We get

$$\int_K [\mathrm{tr}\,\mu(x^{-1})T](\mu(x)w, \tau) = c(\mu)(Tw, \tau). \tag{2.14)(b}$$

Fix another $v \in V_\mu$ and $\xi \in V_\mu^*$, and apply this equation to the endomorphism $T_{v,\xi}$ defined in (2.9)(f). We get

$$\int_K (\mu(x^{-1})v, \xi)(\mu(x)w, \tau) = c(\mu)(w, \xi)(v, \tau). \tag{2.14(c)}$$

This equality is a version of the Schur orthogonality relations ([Knapp], Corollary 1.10; the translation is explained in [KV], (1.22b).) To compute $c(\mu)$, choose a basis $v_1, \cdots, v_n$ of V_μ, and let $\xi_1, \cdots, \xi_n$ be the dual basis of V_μ^*. We apply (2.14)(c) with $v = v_j$, $\xi = \xi_i$, $w = v_k$, and $\tau = \xi_j$, and sum over j, obtaining

$$\int_K \sum_j (\mu(x^{-1})v_j, \xi_i)(\mu(x)v_k, \xi_j) = c(\mu)(v_k, \xi_i) \sum_j (v_j, \xi_j) = c(\mu)\delta_{i,k} \dim V_\mu.$$

Now the number (Tv_j, ξ_i) is nothing but the (i, j) entry of the matrix of T in the basis (v_i). It follows that the integrand on the left is the (i, k) entry of the matrix of $\mu(k^{-1})\mu(k) = \mathrm{Id}$, which is the Kronecker delta δ_{ik}. Because the integrand is constant, the equation reads

$$\mathrm{Vol}(K)\delta_{i,k} = c(\mu)\delta_{i,k} \dim V_\mu.$$

It follows that $c(\mu) = \mathrm{Vol}(K)/\dim V_\mu$, as we wished to show. This completes the proof of (4).

Here is the strategy to complete the argument. First, I will prove the surjectivity of the inverse Fourier transform map in (3). The surjectivity of the Fourier transform (2) follows from (4), and we already know that the map in (1) is surjective. Once all the maps are known to be surjective, their injectivity follows from (4).

To prove the surjectivity of the inverse Fourier transform, it suffices to fix an irreducible representation μ of K, and prove surjectivity to the subspace

$$C(K)_\mu = \{f \in C(K) \mid \lambda(K)f \text{ spans a sum of copies of } V_\mu.\}.$$

The reason is that complete reducibility of finite-dimensional representations of K guarantees that

$$C(K)_F = \sum_{\mu \in \widehat{K}} C(K)_\mu.$$

Now $C(K)_\mu$ is a direct sum of copies of V_μ. It suffices to prove that each summand is in the image of the inverse Fourier transform; so we may fix an intertwining operator

$$\Phi\colon V_\mu \to C(K)_\mu$$

from μ to λ, and prove that the image of Φ is contained in the image of the inverse Fourier transform. Define a linear functional ξ on V_μ by evaluation of the corresponding functions at the identity of K:

$$(v, \xi) = \Phi(v)(1).$$

Then the matrix coefficient corresponding to v and ξ is

$$f^\mu_{v,\xi}(x) = (\mu(x^{-1})v, \xi) = \Phi(\mu(x^{-1})v)(1) = [\lambda(x^{-1})\Phi(v)](1) = \Phi(v)(x).$$

(The third equality is the fact that Φ intertwines μ and λ.) That is, $\Phi(v)$ is equal to the matrix coefficient $f^{\mu}_{v,\xi}$. This proves that $\Phi(v)$ is in the image of the inverse Fourier transform, as we wished to show. □

Theorem 2.13 has a wide variety of important corollaries and reformulations. For one, recall that the *character* of a finite-dimensional representation μ of K is by definition the continuous function

$$\Theta_\mu(k) = \operatorname{tr}\mu(k). \tag{2.15)(a}$$

The trace of the transpose of an operator is equal to the trace of the operator; so it follows that

$$\Theta_{\mu^*}(k) = \operatorname{tr}\mu(k^{-1}). \tag{2.15)(b}$$

In light of the definition of the inverse Fourier transform in Definition 2.10, this can be written as

$$f_{\mathrm{Id}(\mu)} = \Theta_{\mu^*}. \tag{2.15)(c}$$

Here $\mathrm{Id}(\mu)$ is the identity operator on V_μ.

Corollary 2.16. *Suppose K is a compact group, dx is a Haar measure on K, and μ and μ' are two irreducible representations. Then*

$$\frac{\dim V_\mu}{\operatorname{Vol} K}\int_K \Theta_{\mu^*}(x)\mu'(x)dx = \begin{cases} \mathrm{Id}(\mu) & \text{if } \mu' = \mu \\ 0 & \text{if } \mu' \neq \mu. \end{cases}$$

This is immediate from Theorem 2.13(4) and (2.15). What it says is that the measure $\frac{\dim V_\mu}{\operatorname{Vol} K}\Theta_{\mu^*}(x)dx$ maps by the operator-valued Fourier transform to a projection on the μ-isotypic part of any representation: the largest subspace isomorphic to a direct sum of copies of μ.

Here is another perspective on Theorem 2.13. The inverse Fourier transform of Definition 2.10 is a bounded linear map of Banach spaces

$$\operatorname{End} V_\mu \to C(K), \qquad T \mapsto (f_T(x) = \operatorname{tr}[\mu(k^{-1})T]).$$

Such a map has a transpose carrying $C(K)^*$ to $(\operatorname{End} V_\mu)^*$. We already observed (before Definition 2.3) that the dual of $C(K)$ is $\mathcal{M}(K)$. So what is the dual of $\operatorname{End} V_\mu$? The answer is that the pairing

$$(T, A) = \operatorname{tr} TA = \operatorname{tr} AT \qquad (T \in \operatorname{End} V_\mu, A \in \operatorname{End} V_\mu)$$

identifies $(\operatorname{End} V_\mu)^*$ with $\operatorname{End} V_\mu$. (The norms on these spaces do *not* agree, but since they are finite-dimensional this does not matter much.)

Proposition 2.17. *Suppose K is a compact group and μ is a finite-dimensional representation of K. Under the identifications just described, the transpose of the inverse Fourier transform at μ is equal to the operator-valued Fourier transform from $\mathcal{M}(K)$ to $\operatorname{End} V_\mu$.*

Proof. Suppose $T \in \operatorname{End} V_\mu$ and $\delta \in \mathcal{M}(K)$. Then the adjointness statement of the proposition says

$$(\text{inverse Fourier transform of } T, \delta) = (T, \text{Fourier transform of } \delta).$$

Explicitly, this says

$$\int_K \operatorname{tr}[\mu(x^{-1})T]\delta(x) = \operatorname{tr}\left[\left(\int_K \mu(x^{-1})\delta(x)\right)T\right].$$

We can move first the T and then the trace inside the integral on the right, and then this equality is immediate. □

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