Messed up example from class October 2

I have five US coins in my pocket, worth 22 cents. What are they?

First solution. All US coins except the penny have value divisible by five. This implies that the number of pennies I have is either 2 or 7 or 12 ... Since there are only five coins altogether, **the number of pennies must be** 2. That means that I have three non-penny coins worth 20 cents.

The other possible values for a coin are 5, 10, 25, 50, and 100 cents. Since the total value is 20 cents, only nickels and dimes are possible: three nickels and dimes worth 20 cents.

First conclusion is that there can be at most two dimes. If there *are* two, then that's all the value, so no nickels, so only two coins; that's not right, If there are *zero* dimes, then there must be four nickels, which is too many coins; that's not right. The only remaining possibility is **the number of dimes must be** 1. That leaves two coins for the nickels, so **the number of nickels must be** 2.

We've shown that the only *possible* solution is **two pennies, two nickels, one dime**. That really is five coins worth 22 cents, so it's the unique solution.

Second solution. Write x for the number of pennies, y for the number of nickels, and z for the number of dimes. What we're told is (I'll insert row reduction of augmented matrices on the right)

$$\begin{array}{ccccccc} x & + & 5y & + & 10z & = & 22 \\ x & + & y & + & z & = & 5 \end{array} \qquad \left(\begin{array}{ccc|c} 1 & 5 & 10 & 22 \\ 1 & 1 & 1 & 5 \end{array}\right)$$

Subtracting the first equation from the second gives

$$\begin{array}{ccccccc} x & + & 5y & + & 10z & = & 22 \\ & & -4y & + & -9z & = & -17 \end{array} \qquad \left(\begin{array}{ccc|c} 1 & 5 & 10 & 22 \\ 0 & -4 & -9 & -17 \end{array}\right)$$

Multiply the second equation by $-1/4$ to get

$$\begin{array}{ccccccc} x & + & 5y & + & 10z & = & 22 \\ & & y & + & \frac{9}{4}z & = & \frac{17}{4}. \end{array} \qquad \left(\begin{array}{ccc|c} 1 & 5 & 10 & 22 \\ 0 & 1 & \frac{9}{4} & \frac{17}{4} \end{array}\right)$$

Subtract 5 times the second from the first to get

$$\begin{array}{ccccccc} x & & & + & -\frac{5}{4}z & = & \frac{3}{4} \\ & & y & + & \frac{9}{4}z & = & \frac{17}{4} \end{array} \qquad \left(\begin{array}{ccc|c} 1 & 0 & -\frac{5}{4} & \frac{3}{4} \\ 0 & 1 & \frac{9}{4} & \frac{17}{4} \end{array}\right)$$

(the augmented matrix is now reduced row echelon!) which is the same as

$$x = \frac{5}{4}z + \frac{3}{4}, \qquad y = -\frac{9}{4}z + \frac{17}{4}.$$

All these steps are reversible. Conclusion is that z **(the number of dimes) can be anything at all, and then the numbers of nickels and pennies are determined by the last two formulas**: infinitely many solutions.

You should think about how to reconcile these two solutions to the problem. One question: since we wanted solutions with integer numbers of pennies, nickels, and dimes, was it illegal to use rational numbers (remember the division by four!) above?