Unitary representations of reductive groups 1–5

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July 16–20, 2012
Outline

1. Examples and applications of representation theory
   - Fourier series
   - Finite-diml representations
   - Gelfand’s abstract harmonic analysis
   - Quadratic forms and reps of $GL(n)$

2. Examples from automorphic forms
   - Defining automorphic forms
   - Automorphic cohomology

3. Kirillov-Kostant orbit method
   - Commuting algebras
   - Differential operator algebras: how orbit method works
   - Hamiltonian $G$-spaces: how Kostant does the orbit method

4. Classical limit: from group representations to symplectic geometry
   - Associated varieties
   - Deformation quantization
   - Howe’s wavefront set

5. Harish-Chandra’s $(g, K)$-modules
   - Case of $SL(2, \mathbb{R})$
   - Definition of $(g, K)$-modules
   - Harish-Chandra algebraization theorems
How does symmetry inform mathematics (I)?

Example. $\int_{-\pi}^{\pi} \sin^5(t) dt = ?$  Zero!

Principle: group $G$ acts on vector space $V$; decompose $V$ using $G$; study each piece.
Here $G = \{1, -1\}$ acts on $V = \text{functions on } \mathbb{R}$; pieces are even and odd functions.
How does symmetry inform mathematics (II)?

Example. Temp distn $T(t, \theta)$ on hot ring governed by
\[ \frac{\partial T}{\partial t} = c^2 \frac{\partial^2 T}{\partial \theta^2}, \quad T(0, \theta) = T_0(\theta). \]
Too hard for (algebraist) to solve; so look at special initial conditions with rotational (almost) symmetry:
\[ T(0, \theta) = a_0/2 + a_m \cos(m\theta). \]

Diff eqn is symmetric, so hope soln is symmetric:
\[ T(t, \theta) = a_0(t)/2 + a_m(t) \cos(m\theta). \]
Leads to ORDINARY differential equations
\[ \frac{da_0}{dt} = 0, \quad \frac{da_m}{dt} = -c^2 m^2 a_m. \]
These are well-suited to an algebraist:
\[ T(t, \theta) = a_0/2 + a_m e^{-c^2 m^2 t} \cos(m\theta). \]

Generalize: Fourier series expansion of initial temp.

Principle: group $G$ acts on vector space $V$; decompose $V$; study pieces separately. Here $G = \text{rotations of ring}$ acts on $V = \text{functions on ring}$; decomposition is by frequency.
What’s so good about sin and cos?

What’s “cos($m\theta$) is almost rotationally symmetric” mean?

If $f(\theta)$ any function on the circle ($f(\theta + 2\pi) = f(\theta)$), define rotation of $f$ by $\phi$ to be new function $[\rho(\phi)f](\theta) = f(\theta - \phi)$. Rotationally symm. $\equiv \text{unchgd by rotation} \equiv \text{constant}.$

$c_m(\theta) \equiv \text{def} \cos(m\theta), \quad s_m(\theta) \equiv \text{def} \sin(m\theta).

\[ [\rho(\phi)c_m](\theta) = c_m(\theta - \phi) = \cos(m\theta - m\phi) = \cos(m\theta)\cos(m\phi) + \sin(m\theta)\sin(m\phi). \]

Rotation of $c_m$ is a linear combination of $c_m$ and $s_m$: “almost rotationally symmetric.”

Similar calculation for sin shows that

\[ \rho(\phi) \begin{pmatrix} c_m \\ s_m \end{pmatrix} = \begin{pmatrix} \cos(m\phi) & \sin(m\phi) \\ -\sin(m\phi) & \cos(m\phi) \end{pmatrix} \begin{pmatrix} c_m \\ s_m \end{pmatrix}. \]

HARD transcendental rotation $\rightsquigarrow$ EASY linear algebra!
In which we meet the hero of our story...

$$\rho(\phi) \begin{pmatrix} c_m \\ s_m \end{pmatrix} = \begin{pmatrix} \cos(m\phi) & \sin(m\phi) \\ -\sin(m\phi) & \cos(m\phi) \end{pmatrix} \begin{pmatrix} c_m \\ s_m \end{pmatrix}.$$ 

Definition

A representation of a group $G$ on a vector space $V$ is a group homomorphism

$$\rho : G \rightarrow GL(V).$$

Equiv: action of $G$ on $V$ by linear transformations.

Equiv (if $V = \mathbb{C}^n$): each $g \in G \leadsto n \times n$ matrix $\rho(g)$,

$$\rho(gh) = \rho(g)\rho(h), \quad \rho(e) = I_n.$$

HARD questions about $G$, (nonlinear) actions $\leadsto$

EASY linear algebra!
How does symmetry inform math (III)?

First two examples involved easy abelian $G$; usually understood without groups.

Fourier series provide a nice basis 
$\{\cos(m\theta), \sin(m\theta)\}$ for functions on the circle $S^1$.

What analogues are possible on the sphere $S^2$?

$G = O(3) =$ group of $3 \times 3$ real orthogonal matrices, the distance-preserving linear transformations of $\mathbb{R}^3$.

$V =$ functions on $S^2$.

Seek small subspaces of $V$ preserved by $O(3)$.

Example. $V_0 = \langle 1 \rangle =$ constant functions; 1-diml.

Example. $V_1 = \langle x, y, z \rangle =$ linear functions; 3-diml.

Example. $V_2 = \langle x^2, xy, \ldots, z^2 \rangle =$ quad fns; 6-diml.

Problem: $x^2 + y^2 + z^2 = 1$ on $S^2$: so $V_2 \supset V_0$.

Example. $V_m = \langle x^m, \ldots, z^m \rangle =$ deg $m$ polys; $(m+2\choose 2)$-diml.
Polynomials and the group $O(3)$

$$S(\mathbb{R}^3) = \underbrace{\mathbb{V}_0}_{\text{poly fns, dim=1}} + \underbrace{\mathbb{V}_1}_{\text{linear, dim=3}} + \cdots + \underbrace{\mathbb{V}_m}_{\text{degree } m, \text{ dim=}(\frac{m+2}{2})} + \cdots$$

Want to understand restriction of these functions to

$$S^2 = \{(x, y, z) \mid r^2 = 1\} \quad (r^2 = x^2 + y^2 + z^2).$$

**Algebraic geometry point of view ($Q$ for quotient):**

nice fns on $S^2 =_{\text{def}} Q(S^2) = S(\mathbb{R}^3)/\langle r^2 - 1 \rangle$.

To study polynomials with finite-dimensional linear algebra, use the increasing filtration $S^{\leq m}(\mathbb{R}^3)$; get

$$Q^{\leq m}(S^2) = S^{\leq m}(\mathbb{R}^3)/(r^2 - 1)S^{\leq m-2}(\mathbb{R}^3).$$

$$S^{\leq m}(\mathbb{R}^3)/S^{\leq m-1}(\mathbb{R}^3) \simeq V_m,$$

$$Q^{\leq m}(S^2)/Q^{\leq m-1}(S^2) \simeq V_m/(r^2)V_{m-2}.$$

$O(3)$ has rep on $V_m/r^2V_{m-2}$, $\dim = (\frac{m+2}{2}) - (\frac{m}{2}) = 2m + 1$; sum over $m$ gives all (polynomial) fns on $S^2$. 
Polynomials and the group $O(3)$ (reprise)

$$S(\mathbb{R}^3) = V_0 + V_1 + \cdots + V_m + \cdots$$

poly fns \hspace{1cm} \text{constants} \hspace{1cm} \text{linear} \hspace{1cm} \text{degree } m

Want to understand restriction of these functions to $S^2$.

Analysis point of view $\Delta = \partial^2/\partial x^2 + \partial^2/\partial y^2 + \partial^2/\partial z^2$.

nice fns on $S^2 = \text{initial conditions for diff eq } \Delta F = 0$. 

$$V_{m-2} \xrightarrow{r^2} V_m; \quad H_m = \text{def ker}(\Delta | V_m).$$

Proposition

$H_m$ is a complement for $r^2 V_{m-2}$ in $V_m$. Consequently

$$V_m / r^2 V_{m-2} \simeq H_m, \quad (O(3) \text{ rep of dim } = 2m + 1).$$

$$V_m = H_m \oplus r^2 H_{m-2} \oplus r^4 H_{m-4} + \cdots.$$ 

functions on $S^2 \simeq H_0 \oplus H_1 \oplus H_2 \oplus \cdots$
Fourier series on $S^2$

Abstract representation theory: group $O(3)$ has two irr repns of each odd dim $2m + 1$, namely

$$H_m = \text{harmonic polys of deg } m \cong V_m/r^2 V_{m-2},$$
and $H_m \otimes \epsilon$; here

$$\epsilon : O(3) \to \{\pm 1\} \subset GL(1), \quad \text{sgn}(g) =_{\text{def}} \text{sgn} (\det(g)).$$

Schur’s lemma: any invariant Hermitian pairing

$$\langle , \rangle : E \times F \to \mathbb{C}$$

between distinct irreducible representations of a compact group $G$ must be zero. Consequence:

subspaces $H_m \subset L^2(S^2)$ are orthogonal.

Stone-Weierstrass: span$(H_m)$ dense in $L^2(S^2)$.

Proposition

$L^2(S^2)$ is Hilbert space sum of the $2m + 1$-diml subspaces $H_m$ of harmonic polys of degree $m$.

$$f \in L^2(S^2) \to f_m \in H_m, \quad f = \sum_{m=0}^{\infty} f_m.$$ 

Fourier coeff $f_m$ in $2m + 1$-diml $O(3)$ rep.
Gelfand’s abstract harmonic analysis

Topological grp $G$ acts on $X$, have questions about $X$.

Step 1. Attach to $X$ Hilbert space $\mathcal{H}$ (e.g. $L^2(X)$). Questions about $X \rightsquigarrow$ questions about $\mathcal{H}$.

Step 2. Find finest $G$-equivariant decom $\mathcal{H} = \bigoplus_\alpha \mathcal{H}_\alpha$. Questions about $\mathcal{H} \rightsquigarrow$ questions about each $\mathcal{H}_\alpha$.

Each $\mathcal{H}_\alpha$ is irreducible unitary representation of $G$: indecomposable action of $G$ on a Hilbert space.

Step 3. Understand $\hat{G}_u =$ all irreducible unitary representations of $G$: unitary dual problem.

Step 4. Answers about irr reps $\rightsquigarrow$ answers about $X$.

Topic for these lectures: Step 3 for Lie group $G$. Mackey theory (normal subgps) $\rightsquigarrow$ case $G$ reductive.
Making everything noncompact

Examples so far have compact spaces, groups.

\[ D = \text{pos def quad forms in } n \text{ vars} \]
\[ = n \times n \text{ real symm matrices, eigenvalues } > 0 \]
\[ = GL(n, \mathbb{R})/O(n). \]

(invertible \( n \times n \) real matrices modulo subgroup of orthogonal matrices.

\( GL(n, \mathbb{R}) \) acts on \( D \) by change of variables. In matrix realization, \( g \cdot A = gA^t g \). Action is transitive; isotropy group at \( I_n \) is \( O(n) \).

\( C(D) = \text{cont fns on } D, \quad [\lambda(g)f](x) = f(g^{-1} \cdot x) \quad (g \in GL(n, \mathbb{R})); \)

inf-diml rep of \( G \leftrightarrow \) action of \( G \) on \( D \).

Seek (minimal = irreducible) \( GL(n, \mathbb{R})\)-invt subspaces inside \( C(D) \), use them to “decompose” \( L^2(D) \).

(\( V, \rho \)) any rep of \( G = GL(n, \mathbb{R}) \); write \( K = O(n) \).

\[ T \in \text{Hom}_G(V, C(D)) \cong \text{Hom}_K(V, \mathbb{C}) = K\text{-fixed lin fnls on } V \ni \tau, \]
\[ [T(v)](gK) = \tau(\rho(g^{-1}v)). \]
Study $D$ by representation theory

$G = GL(n, \mathbb{R}), \quad K = O(n)$

$D = \text{positive definite quadratic forms}$

$\text{Hom}_G(V, C(D)) \simeq K$-fixed linear functionals on $V$.

So seek to construct (irreducible) reps of $G$ having nonzero $K$-fixed linear functionals.

Idea from Borel-Weil theorem for compact groups:

irr repns $\leftrightarrow$ secs of line bdles on flag mflds.

Complete flag in $m$-diml $E$ is chain of subspaces

$\mathcal{F} = \{0 = F_0 \subset F_1 \subset \cdots \subset F_m = E\}, \quad \dim F_i = i.$

Define $X(\mathbb{R}) = \text{complete flags in } \mathbb{R}^n$. Group $G$ acts transitively on flags. Base point of $X(\mathbb{R})$ is std flag

$\mathcal{F}^0 = \{\mathbb{R}^0 \subset \mathbb{R}^1 \subset \cdots \subset \mathbb{R}^n\}, \quad G^{\mathcal{F}^0} = B,$

$B$ group of upper triangular matrices. Hence $X(\mathbb{R}) \simeq G/B$.

Get rep of $G$ on $V = C(X(\mathbb{R}))$ (functions on flags); has $K$-fixed lin fnl $\tau = \text{integration over } X(\mathbb{R})$. Get embedding

$T: V \hookrightarrow C(D), \quad [Tv](gK) = \int_{x \in X(\mathbb{R})} v(g \cdot x)\,dx.$
Study $D$ by rep theory (continued)

$$G = GL(n, \mathbb{R}), \quad K = O(n), \quad B = \text{upper } \Delta$$

$$D = \text{pos def quad forms } \simeq G/K,$$

$$X(\mathbb{R}) = \text{complete flags in } \mathbb{R}^n \simeq G/B$$

Found embedding

$$T: C(X(\mathbb{R})) \hookrightarrow C(D), \quad [Tv](gK) = \int_{x \in X(\mathbb{R})} v(g \cdot x) \, dx.$$

To generalize, use $G$-eqvt real line bdle $\mathcal{L}_i$ on $X(\mathbb{R})$, $1 \leq i \leq n$; fiber at $\mathcal{F}$ is $F_i/F_{i-1}$.

$$\mathbb{R}^\infty \ni t \mapsto |t|^\nu \text{ sgn}(t)^\epsilon \in \mathbb{C}^\infty \text{ (any } \nu \in \mathbb{C}, \epsilon \in \mathbb{Z}/2\mathbb{Z});$$

Similarly get $G$-eqvt cplx line bdle $\mathcal{L}^{\nu, \epsilon} = \mathcal{L}_1^{\nu_1, \epsilon_1} \otimes \cdots \otimes \mathcal{L}_n^{\nu_n, \epsilon_n}.$

$$V^{\nu, \epsilon} = C(X(\mathbb{R}), \mathcal{L}^{\nu, \epsilon}) = \text{continuous sections of } \mathcal{L}^{\nu, \epsilon}$$

family of reps $\rho^{\nu, \epsilon}$ of $G$: index $n$ cplx numbers, $n$ “parities.”

This is what “all” reps of “all” $G$ look like; study more!

Case all $\epsilon_i = 0$: can make sense of

$$T^{\nu}: V^{\nu, 0} \rightarrow C(D), \quad [T^{\nu} v](gK) = \int_{x \in X(\mathbb{R})} v(g \cdot x) \, dx.$$
Study $D$ directly

$$G = GL(n, \mathbb{R}), \quad K = O(n)$$

$D$ = positive definite quadratic forms.

Seek (minimal = irreducible) $GL(n, \mathbb{R})$-invt subspaces inside $C(D)$, use them to “decompose” $L^2(D)$.

If $G$ acts on functions, how do you find invt subspaces?

Look at this in third lecture. For now, two ideas...

Can scale pos def quad forms (mult by nonzero pos real):

$$C(D) \supset C^{\lambda_1}(D) = \text{fns homog of degree } \lambda_1 \in \mathbb{C}.$$

$$= \{ f \in C(D) \mid f(tx) = t^{\lambda_1} f(x) \quad (t \in \mathbb{R}^+, x \in D) \}$$

$$= \{ f \in C(D) \mid \Delta_1 f = \lambda_1 f \},$$

$\Delta_1 =$ Euler degree operator $= \sum_j x_j \partial/\partial x_j$.

$D$ has $G$-invt Riemannian structure and therefore Laplace operator $\Delta_2$ commuting with $G$.

$$C(D) \supset C^{\lambda_2}(D) = \lambda_2\text{-eigenspace of } \Delta_2$$

$$= \{ f \in C(D) \mid \Delta_2 f = \lambda_2 f \quad (\lambda_2 \in \mathbb{C}) \}.$$
Study $D$ directly (continued)

$$G = GL(n, \mathbb{R}), \quad K = O(n)$$

$D$ = positive definite quadratic forms.

Seek (minimal = irreducible) $GL(n, \mathbb{R})$-invt subspaces.

So far: found eigenspaces of two $G$-invt diff ops (Euler degree op $\Delta_1$, Laplace op $\Delta_2$)

**Theorem (Harish-Chandra, Helgason)**

Algebra $D^G$ of $G$-invt diff ops on $D$ is a (comm) poly ring, gens $\{\Delta_1, \Delta_2, \ldots, \Delta_n\}$, $\deg(\Delta_j) = j$.

Get nice $G$-invt spaces of (analytic) functions

$$C(D) \ni C^\lambda(D) = \text{joint eigenspace of all } \Delta_j$$

$$= \{ f \in C(D) \mid \Delta_j f = \lambda_j f \quad (1 \leq j \leq n) \}.$$ 

Relation to rep-theoretic approach: had

$$T^\nu : V^\nu,0 \rightarrow C(D), \quad [T^\nu]v(gK) = \int_{x \in X(\mathbb{R})} v(g \cdot x) \, dx$$

Here $V^\nu = \text{secs of bundle on flag variety } X(\mathbb{R})$; each $V^\nu$ maps to one eigenspace $\lambda(\nu)$. 

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1. Why representations?
   - Fourier series
   - Finite-diml representations
   - Abstract harmonic analysis
   - Quadratic forms

2. Automorphic forms
   - Defining automorphic forms
   - Automorphic cohomology

3. Orbit method
   - Commuting algebras
   - Differential operator algebras
   - Hamiltonian $G$-spaces

4. Classical limit
   - Associated varieties
   - Deformation quantization
   - Howe's wavefront set

5. $(g, K)$-modules
   - Case of $SL(2, \mathbb{R})$
   - Definition of $(g, K)$-modules
   - Harish-Chandra algebraization theorems
What’s so great about automorphic forms?

**Arithmetic questions** (about ratl solns of poly eqns) hard: lack tools from analysis and geometry).

Cure: embed arithmetic questions in **real** ones... 

Arithmetic: cardinality of \( \{(p, q) \in \mathbb{Z}^2 \mid p^2 + q^2 \leq N\} \)?

Geom: area of \( \{(p, q) \in \mathbb{R}^2 \mid p^2 + q^2 \leq N\} \)? Ans: \( N\pi \).

Conclusion: answer to arithmetic question is “\( N\pi + \text{small error} \).”

Error \( O(N^{131/416+\epsilon}) \) (Huxley 2003); conjecturally \( N^{1/4+\epsilon} \).

Similarly: counting solns of arithmetic eqns mod \( p^n \) ↔ analytic/geometric problems over \( \mathbb{Q}_p \).

Model example: relationship among \( \mathbb{Z}, \mathbb{R}, \text{circle} \).

**Algebraic/counting** problems live on \( \mathbb{Z} \); **analysis** lives on \( \mathbb{R} \); **geometry** lives on circle \( \mathbb{R}/\mathbb{Z} \).

Automorphic forms provide parallel interaction among arithmetic, analysis, geometry.
What's so great about automorphic forms

Theorem

Write $\mathbb{A} = \mathbb{R} \times \prod'_p \mathbb{Q}_p$ (restricted product). Then $\mathbb{A}$ is locally compact topological ring containing $\mathbb{Q}$ as a discrete subring, and $\mathbb{A}/\mathbb{Q}$ is compact.

Corollary

1. $GL(n, \mathbb{A}) = GL(n, \mathbb{R}) \times \prod'_p GL(n, \mathbb{Q}_p)$ is loc cpt grp.
2. $GL(n, \mathbb{Q})$ is a discrete subgroup.
3. Quotient space $GL(n, \mathbb{A})/GL(n, \mathbb{Q})$ is nearly compact.

Conclusion: the space $GL(n, \mathbb{Q}) \backslash GL(n, \mathbb{A})$ is a convenient place to relate arithmetic and analytic questions.

$\mathcal{A}(n)$ = automorphic forms on $GL(n) =$ functions on $GL(n, \mathbb{Q}) \backslash GL(n, \mathbb{A})$ (+ technical growth conds).

Vector space $\mathcal{A}(n)$ is a representation of $GL(n, \mathbb{A})$.

Irr constituents of $\mathcal{A}(n)$ are automorphic representations; carry information about arithmetic.
What’s that mean really???

\[ K = O(n) \times \prod_p GL(n, \mathbb{Z}_p) \] is compact subgroup of
\[ GL(n, \mathbb{A}) = GL(n, \mathbb{R}) \times \prod'_p GL(n, \mathbb{Q}_p). \]

Since representation theory for compact groups is nice, can look only at “almost K-invt” automorphic forms.

\[ \mathcal{A}(n)^K = \text{fns on } GL(n, \mathbb{Q}) \backslash GL(n, \mathbb{A}) / K. \]

Easy:

\[ GL(n, \mathbb{Q}) \backslash GL(n, \mathbb{A}) / K \supset GL(n, \mathbb{Z}) \backslash GL(n, \mathbb{R}) / O(n) \]

\[ = GL(n, \mathbb{Z}) \backslash D \]

\[ = GL(n, \mathbb{Z}) \backslash \text{pos def forms} \]

\[ = \{(\text{rk } n \text{ lattice, } \mathbb{R}-\text{val pos def form})\} / \sim \]

Conclusion: automorphic form on \( GL(n) \approx \) fn on isom classes of \([\text{rank } n \text{ lattice w pos def } \mathbb{R}-\text{valued form}]. \)

More general automorphic forms:

\[ GL(n, \mathbb{Z}_p) \rightsquigarrow \text{open subgp} \quad GL(n, \mathbb{Z}) \rightsquigarrow \text{cong subgp } \Gamma \]

\[ O(n)-\text{inv} \rightsquigarrow \text{rep } E \text{ of } O(n) \quad \text{fns on } \Gamma \backslash D \rightsquigarrow \text{secs of } \mathcal{E} \rightarrow \Gamma \backslash D \]

\( G \) reductive group defined over \( \mathbb{Q} \): replace \( GL(n), \) by \( G(. \).
What representation theory can tell you (I)

Automorphic forms $\mathcal{A}(n)$ for $GL(n)$...

Make “decomposition” as in Gelfand’s abstract program

$$\mathcal{A}(n) = \int_{\pi \in GL(n, \mathbb{A})_u} V_{\pi} \otimes M(\pi, \mathcal{A}(n)).$$

$V_{\pi} = \text{rep space of } \pi, \ M = \text{multiplicity space}.$

Done by Langlands (1965).

$$K\text{-invt aut forms} = \mathcal{A}(n)^K$$

$$= \int_{\pi \in GL(n, \mathbb{A})_u} V_{\pi}^K \otimes M(\pi, \mathcal{A}(n)).$$

Knowing which unitary reps $\pi$ can have $V_{\pi}^K \neq 0$ restricts $K$-invt automorphic forms.

Knowing which unitary reps of $GL(n, \mathbb{R})$ can have $O(n)$-fixed vectors restricts $L^2(GL(n, \mathbb{Z}) \backslash D)$.

Questions answered (for $GL(n)$) by DV, Tadić in 1980s.
What representation theory can tell you (II)

Example. $X$ compact (arithmetic) locally symmetric manifold of dim 128; $\dim (H^{28}(X, \mathbb{C})) = ?$ Eight!

Same as $H^{28}$ for compact globally symmetric space.

Generalize: $X = \Gamma \backslash G/K$, $H^p(X, \mathbb{C}) = H^p_{\text{cont}}(G, L^2(\Gamma \backslash G))$. Decompose $L^2$:

$$L^2(\Gamma \backslash G) = \sum_{\pi \text{ irr rep of } G} m_\pi(\Gamma) \mathcal{H}_\pi \quad (m_\pi = \dim \text{ of some aut forms})$$

Deduce $H^p(X, \mathbb{C}) = \sum_{\pi} m_\pi(\Gamma) \cdot H^p_{\text{cont}}(G, \mathcal{H}_\pi)$.

General principle: group $G$ acts on vector space $V$; decompose $V$; study pieces separately.
Time for something serious

Can’t emphasize enough how important this idea is.
What the orbit method does

Gelfand’s program says that to better understand problems involving Lie group $G$, should understand $\hat{G}_u$, the set of equiv classes of irr unitary reps $\pi$ of $G$.

Such $\pi$ is homomorphism of $G$ into group of unitary operators on (usually $\infty$-diml) Hilbert space $\mathcal{H}_\pi$: seems much more complicated than $G$; so what have we gained?

How should we think of an irr unitary representation?

Kirillov-Kostant idea: philosophy of coadjoint orbits. . .

irr unitary rep $\leftrightarrow$ coadjoint orbit,

orbit of $G$ on dual vector space $g^*$ of $g_0 = \text{Lie}(G)$.

Case of $GL(n)$: says unitary rep is more or less a conj class of $n \times n$ matrices.

Will explain what this statement means, why it is reasonable, and how one can try to prove it.
Decomposing a representation

Given: interesting operators $\mathcal{A}$ on Hilbert space $\mathcal{H}$.
Goal: decompose $\mathcal{H}$ in $\mathcal{A}$-invt way.

Finite-dimensional case:
$V/\mathbb{C}$ fin-diml, $\mathcal{A} \subset \text{End}(V)$ cplx semisimple algebra.

Classical (Wedderburn) structure theorem:
$W_1, \ldots, W_r$ list of all simple $\mathcal{A}$-modules; then

$$\mathcal{A} \simeq \text{End}(W_1) \times \cdots \times \text{End}(W_r) \quad V \simeq m_1 W_1 + \cdots + m_r W_r.$$ 

Positive integer $m_i$ is multiplicity of $W_i$ in $V$.

Slicker version: define multiplicity space $M_i = \text{Hom}_\mathcal{A}(W_i, V)$; then $m_i = \dim M_i$, and

$$V \simeq M_1 \otimes W_1 + \cdots + M_r \otimes W_r.$$ 

Slickest version: COMMUTING ALGEBRAS...
Commuting algebras and all that

Theorem

\( A = \text{semisimple algebra of ops on fin-diml } V \text{ as above; define } Z = \text{Cent}_{\text{End}(V)}(A), \text{second semisimple alg of ops on } V. \)

1. Relation between \( A \) and \( Z \) is symmetric:
   \[ A = \text{Cent}_{\text{End}(V)}(Z). \]

2. There is a natural bijection between irr modules \( W_i \) for \( A \) and irr modules \( M_i \) for \( Z \), given by
   \[ M_i \cong \text{Hom}_A(W_i, V), \quad W_i \cong \text{Hom}_Z(M_i, V). \]

3. \( V \cong \sum_i M_i \otimes W_i \) as a module for \( A \times Z \).

Example 1: finite \( G \) acts left and right on \( V = \mathbb{C}[G] \).

Example 2: \( S_n \) and \( GL(E) \) act on \( V = T^n(E) \).

But those are stories for other days...
A version for Lie algebras

Just to show that commuting algebra idea can be made to work... $\mathfrak{g} \supset \mathfrak{h}$ reductive in $\mathfrak{g}$.

$\mathcal{A} = \text{def } U(\mathfrak{h})$, $\mathcal{Z} = \text{Cent}_{U(\mathfrak{g})}(\mathcal{A}) = U(\mathfrak{g})^\mathfrak{h}$.

Fix $V = U(\mathfrak{g})$-module. For $(\mu, E_\mu)$ fin diml $\mathfrak{h}$-irr, set

$$M_\mu = \text{Hom}_\mathcal{A}(E_\mu, V) = \text{Hom}_\mathfrak{h}(E_\mu, V); \quad \text{then}$$

$$M_\mu \otimes E_\mu \hookrightarrow V \quad (\text{all copies of } \mu \text{ in } V);$$

and $M_\mu$ is $\mathcal{Z}$-module.

Theorem (Lepowsky-McCollum)

Suppose $V$ irr for $\mathfrak{g}$, and action of $\mathfrak{h}$ locally finite. Then

$$V = \sum_{\mu \text{ for } \mathfrak{h}} M_\mu \otimes E_\mu.$$ 

Each $M_\mu$ is an irreducible module for $\mathcal{Z}$; and $M_\mu$ determines $\mu$ and $V$. 

Infinite-dimensional representations

Need framework to study ops on inf-diml \( V \).

Dictionary

<table>
<thead>
<tr>
<th>Fin-diml</th>
<th>( \Leftrightarrow )</th>
<th>Inf-diml</th>
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<tbody>
<tr>
<td>finite-diml ( V )</td>
<td>( \Leftrightarrow )</td>
<td>( C^\infty(M) )</td>
</tr>
<tr>
<td>repn of ( G ) on ( V )</td>
<td>( \Leftrightarrow )</td>
<td>action of ( G ) on ( M )</td>
</tr>
<tr>
<td>( \text{End}(V) )</td>
<td>( \Leftrightarrow )</td>
<td>( \text{Diff}(M) )</td>
</tr>
<tr>
<td>( A = \text{im}(\mathbb{C}[G]) \subset \text{End}(V) )</td>
<td>( \Leftrightarrow )</td>
<td>( A = \text{im}(U(\mathfrak{g})) \subset \text{Diff}(M) )</td>
</tr>
<tr>
<td>( Z = \text{Cent}_{\text{End}(V)}(A) )</td>
<td>( \Leftrightarrow )</td>
<td>( Z = G)-invt diff ops</td>
</tr>
</tbody>
</table>

Suggests: \( G \)-irr \( V \subset C^\infty(M) \) \( \leftrightarrow \) simple modules \( E \) for \( \text{Diff}(M)^G \), \( V \leftrightarrow \text{Hom}_{\text{Diff}(M)^G}(E, C^\infty(M)) \).

Suggests: \( G \) action on \( C^\infty(M) \) irr \( \leftrightarrow \) \( \text{Diff}(M)^G = \mathbb{C} \).

Not always true, but a good place to start.

Which differential operators commute with \( G \)?
Differential operators and symbols

\[ \text{Diff}_n(M) = \text{diff operators of order} \leq n. \]

Increasing filtration, \((\text{Diff}_p)(\text{Diff}_q) \subset \text{Diff}_{p+q}.\)

**Theorem (Symbol calculus)**

1. *There is an isomorphism of graded algebras*

   \[ \sigma : \text{gr Diff}(M) \to \text{Poly}(T^*(M)) \]

   *to fns on \(T^*(M)\) that are polynomial in fibers.*

2. \(\sigma_n : \text{Diff}_n(M)/\text{Diff}_{n-1}(M) \to \text{Poly}^n(T^*(M)).\)

3. *Commutator of diff ops \(\sim\) Poisson bracket \(\{,\}\) on \(T^*(M):\) for \(D \in \text{Diff}_p(M), D' \in \text{Diff}_q(M),\)

   \[ \sigma_{p+q-1}([D, D']) = \{\sigma_p(D), \sigma_q(D')\}. \]

Diff ops comm with \(G \sim \sim \) symbols Poisson-comm with \(g.\)

\(\sim\sim : \to \) is true, and \(\sim \leq \) closer than you’d think.

Orig question which diff ops commute with \(G?\) becomes which functions on \(T^*(M)\) Poisson-commute with \(g?\)
Poisson structure and Lie group actions

To find fns on $T^*(M)$ Poisson-comm w $\mathfrak{g}$, generalize... 

Poisson manifold $X$ has Lie bracket $\{,\}$ on $C^\infty(M)$, such that $\{f, \cdot\}$ is a derivation of $C^\infty(M)$. Poisson bracket on $T^*(M)$ is an example.

Bracket with $f \mapsto \xi_f \in \text{Vect}(X)$: $\xi_f(g) = \{f, g\}$.

Vector flds $\xi_f$ called Hamiltonian; preserve $\{,\}$. Map $C^\infty(X) \to \text{Vect}(X)$, $f \mapsto \xi_f$ is Lie alg homomorphism.

$G$ acts on mfld $X \mapsto$ Lie alg hom $\mathfrak{g} \to \text{Vect}(X)$, $Y \mapsto \xi_Y$.

Poisson $X$ is Hamiltonian $G$-space if Lie alg action lifts

$$
\begin{array}{ccc}
C^\infty(X, \mathbb{R}) & & f_Y \\
\uparrow & & \downarrow \\
\mathfrak{g}_0 & \rightarrow & \text{Vect}(X) \\
\end{array}
\begin{array}{ccc}
Y & \mapsto & \xi_Y \\
\uparrow & & \downarrow \\
\mathfrak{g}_0 & \rightarrow & C^\infty(X, \mathbb{R})
\end{array}
$$

A linear map $\mathfrak{g}_0 \to C^\infty(X, \mathbb{R})$ is the same thing as a smooth moment map $\mu: X \to \mathfrak{g}_0^*$. 

---

1. Why representations?
   Fourier series
   Finite-diml representations
   Abstract harmonic analysis
   Quadratic forms

2. Automorphic forms
   Defining automorphic forms
   Automorphic cohomology

3. Orbit method
   Commuting algebras
   Differential operator algebras
   Hamiltonian G-spaces

4. Classical limit
   Associated varieties
   Deformation quantization
   Howe’s wavefront set

5. $(\mathfrak{g}, K)$-modules
   Case of $SL(2, \mathbb{R})$
   Definition of $(\mathfrak{g}, K)$-modules
   Harish-Chandra algebraization theorems
Poisson structure and invt diff operators

\( X \) Hamiltonian \( G \)-space, moment map \( \mu : X \to g^* \)
G-eqvt map of Poisson mflds,
\[
f_Y(x) = \langle \mu(x), Y \rangle \quad (Y \in g_0, x \in X).
\]
\( f \in C^\infty(X) \) Poisson-commutes with \( g_0 \)
\[
\iff \xi_Y f = 0, \quad (Y \in g_0)
\]
\( \iff f \) constant on \( G \) orbits on \( X \).

Only \( C \) Poisson-comm with \( g_0 \iff \) dense orbit on \( X \).

Proves: dense orbit on \( T^*(M) \implies \text{Diff}(M)^G = C \).

Suggests: \( G \) irr on \( C^\infty(M) \iff \) dense orbit on \( T^*(M) \).

Suggests to a visionary: Irr reps of \( G \) correspond to homogeneous Hamiltonian \( G \)-spaces.
Recall: Hamiltonian $G$-space $X$ comes with ($G$-equivariant) moment map $\mu : X \rightarrow g_0^*$. 
Kostant’s theorem: homogeneous Hamiltonian $G$-space = covering of $G$-orbit on $g_0^*$. 
Recall: commuting algebra formalism for diff operators suggests irreducible representations $\rightarrow$ homogeneous Hamiltonian $G$-spaces.

Kirillov-Kostant philosophy of coadjt orbits suggests
\[
\{\text{irr unitary reps of } G\} = \hat{G}_U \leftrightarrow g_0^*/G. \quad (\star)
\]

MORE PRECISELY... restrict right side to “admissible” orbits (integrality cond). Expect to find “almost all” of $\hat{G}_U$: enough for interesting harmonic analysis.
Evidence for orbit method

With the caveat about restricting to admissible orbits... \[ \hat{G}_u \leftrightarrow g^*/G. \]  

(⋆) is true for $G$ simply conn nilpotent (Kirillov).

(⋆) is true for $G$ type I solvable (Auslander-Kostant).

(⋆) for algebraic $G$ reduces to reductive $G$ (Duflo).

Case of reductive $G$ is still open.

Actually (⋆) is false for connected nonabelian reductive $G$.

But there are still theorems close to (⋆).

Two ways to do repn theory for reductive $G$:

1. start with coadjt orbit, look for repn. Hard: Lecture 5.
2. start with repn, look for coadjt orbit. Easy: Lecture 4.

Really need to do both things at once. Having started to do mathematics in the Ford administration, I find this challenging. (Gave up chewing gum at that time.)
“Classical limit” direction of the orbit philosophy asks for a map (irr unitary reps) \( \mapsto \) orbits in \( g_0^* \).

\( V \) rep of complex Lie alg \( g \).

Assume \( V \) is finitely generated: exists fin diml \( V_0 \subset V \) so that \( U(g) V_0 = V \).

Define increasing family of subspaces
\( V_0 \subset V_1 \subset V_2 \subset \cdots, \ V_m = U_m(g) V_0 \).

\( V_m = \text{span of } Y_1 \cdot Y_2 \cdots Y_m' \cdot v_0, \ (v_0 \in V_0, Y_i \in g, m' \leq m) \).

Action of \( g \) gives \( g \times V_m \to V_{m+1}, \ (Y, v_m) \mapsto Y \cdot v_m \), and therefore a well-defined map
\( g \times [V_m/V_{m-1}] \to [V_{m+1}/V_m], \ (Y, v_m + V_{m-1}) \mapsto Y \cdot v_m + V_m \).

Actions of different elts of \( g \) commute; so \( \text{gr } V \) is a graded \( S(g) \)-module generated by the fin-diml subspace \( V_0 \).

**Associated variety** \( \text{Ass}(V) = \text{supp}({\text{gr } V}) \subset g^* \) (defined by commutative algebra).
What's good about Ass($V$)

$V$ fin gen $/U(g), V_m = U_m(g)V_0$, Ass($V$) = supp(gr($V$)).

Commutative algebra tells you many things:

1. $\dim V_m = p_V(m)$, is a polynomial function of $m$.
2. The degree $d$ of $p_V$ is $\dim($Ass($V$)). Define the Gelfand-Kirillov dimension of $V$ to be $\text{Dim } V = d$.
3. $l_{gr} = \text{def } \text{Ann}(\text{gr}(V)) \subset S(g)$, graded ideal; then $d = \dim(S(g)/l_{gr})$ (Krull dimension).
4. $I = \text{def } \text{Ann}(V) \subset U(g)$ 2-sided ideal; gr $I \subset l_{gr}$, usually $\neq$.

Example. $g = \text{span}(p, q, z), [p, q] = z, [z, p] = [z, q] = 0$.

$V = \mathbb{C}[x], \quad p \cdot f = df/dx, \quad q \cdot f = xf, z \cdot f = f$.

This is (irr) rep of $g$ generated by $V_0 = \mathbb{C}$.

$V_m = \text{polys in } x$ of degree $\leq m, \quad \dim V_m = m + 1$.

$\text{gr } V \simeq \mathbb{C}[x]; \quad p \sim \text{mult by } x; \quad q, z \sim \text{zero}; \quad l_{gr} = \langle q, z \rangle \subset S(g)$.

$I = \langle z - 1 \rangle, \quad U(g)/I \simeq \text{Weyl algebra } \mathbb{C}[d/dx, x], \quad \text{gr } I = \langle z \rangle$.

Ass($V$) = $\{ \lambda \in g^* \mid \lambda(q) = \lambda(z) = 0 \} \subset \text{supp(gr } I) = \{ \lambda \mid \lambda(z) = 0 \}$. 
What’s bad about $\text{Ass}(V)$

For fin gen $M$ over poly alg $S$, $I = \text{Ann}(M) \subset S$,
\[
\text{Dim}(M) = \text{Dim } S/I, \quad \text{supp } M = \text{supp}(I).
\]

For fin gen $V$ over $U(\mathfrak{g})$, $I = \text{Ann}(V)$, $I_{\text{gr}} = \text{Ann}(\text{gr}(V))$,
\[
\text{Dim}(V) = \text{Dim } S(\mathfrak{g})/I_{\text{gr}}, \quad \text{Ass}(V) = \text{supp}(I_{\text{gr}}), \quad \text{but}
\text{gr}(I) \subset I_{\text{gr}}, \quad \text{supp(gr }I) \supset \text{Ass}(V), \quad \text{Dim}(S(\mathfrak{g})/\text{gr }I) \geq \text{Dim}(V);
\]
containments and inequalities generally strict.

Closely related and worse: even if $V$ related to nice rep of $G$,
$\text{Ass}(V)$ rarely preserved by $G$. Some good news...

**Proposition**

$V$ fin gen / $U(\mathfrak{g})$ by $V_0$, $V_0$ preserved by $\mathfrak{h} \subset \mathfrak{g} \implies \text{Ass}(V) \subset (\mathfrak{g}/\mathfrak{h})^*$

stable under coadjt action of $H$.

I 2-sided ideal in $U(\mathfrak{g}) \implies \text{Ass(gr }I) \text{ G-stable}.$

Ideal picture (correct for irr $(\mathfrak{g}, K)$-modules defined *infra*):
\[
V = \text{irr } U(\mathfrak{g}) \text{-module},
\]
\[
I = \text{Ann}(V) = \text{2-sided prim ideal in } U(\mathfrak{g});
\]
\[
\text{Ass}(I) = \text{aff alg Hamilt. G-space, } \text{dim Ass}(I) = 2d;
\]
\[
\text{Ass}(V) = \text{coisotropic subvar of } X, \quad \text{dim Ass}(V) = d.
\]
Deformation quantization and wishful thinking

Here is how orbit method might work for reductive groups.

\( G(\mathbb{R}) = \) real points of conn cplx reductive alg \( G(\mathbb{C}) \).

Start with \( \mathcal{O}_0 \subset \mathfrak{g}_0^* \) coadjoint orbit for \( G(\mathbb{R}) \).

\[ \mathcal{O}(\mathbb{C}) = \text{def} \ G(\mathbb{C}) \cdot \mathcal{O}_0, \quad J_{\mathcal{O}} = \text{ideal of } \mathcal{O}(\mathbb{C}). \]

\( \mathcal{O}_0 \subset \mathcal{O}(\mathbb{R}) \) must be open, but may be proper subset.

Ring of functions \( R_{\mathcal{O}} = S(\mathfrak{g})/J_{\mathcal{O}} \) makes \( \overline{\mathcal{O}}(\mathbb{C}) \) affine alg Poisson variety, Hamiltonian \( G \)-space. (Better: normalize to slightly larger algebra \( R(\mathcal{O}(\mathbb{C})) \).)

Simplify: \( \mathcal{O}(\mathbb{C}) \) nilp; equiv, \( J_{\mathcal{O}} \) and \( R_{\mathcal{O}} \) graded:

\[ R_{\mathcal{O}} = \sum_{p \geq 0} R^p, \quad R^p \cdot R^q \subset R^{p+q}, \quad \{ R^p, R^q \} \subset R^{p+q-1}. \]

\( G \)-eqvt deformation quantization of \( \overline{\mathcal{O}} \) is filtered algebra \( D = \bigcup_{p \geq 0} D_p, \ G(\mathbb{C}) \) action by alg auts, symbol calculus

\[ \sigma_p : D_p/D_{p-1} \xrightarrow{\sim} R^p \]
What deformation quantization looks like

\[ R_O = \sum_{p \geq 0} R^p \] graded ring of fns on cplx nilpotent coadjt orbit, \( D_p \) “corresponding” filtered algebra with \( G(\mathbb{C}) \) action.

Since \( G(\mathbb{C}) \) reductive, can choose \( G(\mathbb{C}) \)-stable complement \( C^p \) for \( D_{p-1} \) in \( D_p \); then \( \sigma_p : C^p \simto R^p \) must be isom, so have \( G(\mathbb{C}) \)-eqvt linear isoms

\[ D_p = \sum_{q \leq p} C^q \xrightarrow{\sigma} \sum_{q \leq p} R^p, \quad D \xrightarrow{\sigma} R. \]

Mult in \( D \) defines via isom \( \sigma \) new assoc product \( m \) on \( R \):

\[ m : R \times R \to R, \quad m(r, s) = \sigma \left( \sigma^{-1}(r) \cdot \sigma^{-1}(s) \right). \]

Filtration on \( D \) implies that for \( r \in R^p, s \in R^q \),

\[ m(r, s) = \sum_{k=0}^{p+q} m_k(r, s), \quad m_k(r, s) \in R^{p+q-k}. \]

**Proposition**

\( G(\mathbb{C}) \)-eqvt deformation quantization of alg \( R_O \) (fns on a cplx nilp coadjt orbit) given by \( G(\mathbb{C}) \)-eqvt bilinear maps

\[ m_k : R^p \times R^q \to R^{p+q-k}, \] subject to \( m_0(r, s) = r \cdot s, \)

\( m_1(r, s) = \{r, s\}, \) and the reqt that \( \sum_{k=0}^{\infty} m_k \) is assoc.

**OPEN PROBLEM: PROVE DEFORMATIONS EXIST.**
Why this is reasonable

$P(\mathbb{C}) \subset G(\mathbb{C})$ parabolic, $M(\mathbb{C}) = G(\mathbb{C})/P(\mathbb{C})$ proj alg. $G(\mathbb{C})$ has unique open orbit $\tilde{O}(\mathbb{C}) \subset T^*M(\mathbb{C})$, which by Kostant must be finite cover of nilp coadjt orbit $O(\mathbb{C})$:

$$\tilde{O}(\mathbb{C}) \subset T^*M(\mathbb{C})$$

$\downarrow \mu \tilde{O} \quad \downarrow \mu$

$O(\mathbb{C}) \subset \overline{O(\mathbb{C})} \subset g^*$

$\mu \tilde{O}$ is finite cover; $\mu$ is proper surjection. Put

$$D = \text{alg diff ops on } M(\mathbb{C}), \quad S = \text{alg fns on } T^*M(\mathbb{C})$$

$$R_{\text{norm}} = \text{alg fns on } O(\mathbb{C}), \quad R = \text{alg fns on } \overline{O(\mathbb{C})}.$$

1. Symbol calculus provides isom $\text{gr } D \xrightarrow{\sigma} S$
2. Restriction provides isom $S \simeq \text{alg fns on } \tilde{O}(\mathbb{C})$.
3. $\mu \tilde{O}$ isom $\Leftrightarrow$ cover triv $\Leftrightarrow$ $\mu$ is birational.
4. Inclusion exhibits $R_{\text{norm}}$ as normalization of $R$.

Conclusion (Borho-Jantzen): $D$ is nice deformation quantization of $O(\mathbb{C}) \Leftrightarrow \mu$ birational with normal image.

Always true for $GL(n)$. 
Simple complex facts

$G(\mathbb{C})$ cplx conn reductive alg, $\mathfrak{g} = \text{Lie}(G(\mathbb{C}))$.

$\mathfrak{h} \subset \mathfrak{b} = \mathfrak{h} + \mathfrak{n} \subset \mathfrak{g}$ Cartan and Borel subalgebras.

$x_s \in \mathfrak{g}$ **semisimple** if following equiv conds hold:

1. $\text{ad}(x_s)$ diagonalizable;
2. $\rho(x_s)$ diagonalizable, all $\rho : G(\mathbb{C}) \to GL(N, \mathbb{C})$ alg.
3. $G(\mathbb{C}) \cdot x_s$ is closed;
4. $G(\mathbb{C}) \cdot x_s$ meets $\mathfrak{h}$.
5. $G(\mathbb{C})^x_s$ is reductive.

$x_n \in \mathfrak{g}$ **nilpotent** if following equiv conds hold:

1. $\text{ad}(x_n)$ nilpotent and $x_n \in [\mathfrak{g}, \mathfrak{g}]$;
2. $\rho(x_n)$ nilpotent, all $\rho : G(\mathbb{C}) \to GL(N, \mathbb{C})$ alg.
3. $G(\mathbb{C}) \cdot x_n$ closed under dilation;
4. $G(\mathbb{C}) \cdot x_n$ meets $\mathfrak{n}$.

**Jordan decomposition**: every $x \in \mathfrak{g}$ is uniquely $x = x_s + x_n$ with $x_s$ semisimple, $x_n$ nilpotent, $[x_s, x_n] = 0$. 
Simple complex dual facts

\(G(\mathbb{C})\) still cplx reductive, \(g^*\) = complex dual space, \(\text{Ad}^*\) coadjoint action of \(G(\mathbb{C})\).

There exists symm \(\text{Ad}\)-invt form on \(g\); equiv, \(g \simeq g^*\), \(\text{Ad} \simeq \text{Ad}^*\). Can use to transfer previous slide to \(g^*\).

**THIS IS ALWAYS A BAD IDEA**: \(g^*\) is different.

\(\lambda_s \in g^*\) *semisimple* if following equiv conds hold:
1. \(G(\mathbb{C}) \cdot \lambda_s\) is closed;
2. \(G(\mathbb{C}) \lambda_s\) is reductive.

\(\lambda_n \in g^*\) *nilpotent* if following equiv conds hold:
1. \(G(\mathbb{C}) \cdot \lambda_n\) closed under dilation;
2. \(\lambda_n\) vanishes on some Borel subalgebra of \(g\).
3. For each \(p \in S(g)^{G(\mathbb{C})}\), \(p(\lambda_n) = p(0)\).

**Jordan decomposition**: every \(\lambda \in g^*\) is uniquely \(\lambda = \lambda_s + \lambda_n\) with \(\lambda_s\) semisimple, \(\lambda_n\) nilpotent, and \(\lambda_s + t\lambda_n \in G(\mathbb{C}) \cdot \lambda\) (all \(t \in \mathbb{C}^\times\)).

**PROBLEM**: extend these lists of equiv conds. Find analogue of Jacobson-Morozov for nilpotents in \(g^*\).
Back to associated varieties

\[ \mathfrak{z}(\mathfrak{g}) = \text{center of } U(\mathfrak{g}); \text{ at first } \mathfrak{g} \text{ is arbitrary.} \]

**Definition**

Rep \((\pi, V)\) of \(\mathfrak{g}\) is *quasisimple* if \(\pi(z) = \text{scalar, all } z \in \mathfrak{z}(\mathfrak{g})\). Alg homomorphism \(\chi_V : \mathfrak{z}(\mathfrak{g}) \to \mathbb{C}\) is the *infinitesimal character of \(V\). Write \(J_V = \ker(\chi_V)\), maximal ideal in \(\mathfrak{z}(\mathfrak{g})\).

**Easy fact:** any irr \(V\) is quasisimple, so \(I_V = \text{Ann}(V) \supset J_V\), so \(\text{gr } I_V \supset \text{gr } J_V\).

**Another easy fact:** \(\text{gr } \mathfrak{z}(\mathfrak{g}) = S(\mathfrak{g})^{G(\mathbb{C})}\).

So \(\text{gr } J_V\) is graded maximal ideal in \(S(\mathfrak{g})^{G(\mathbb{C})}\), so

\[ \text{gr } I_V \supset \text{gr } J_V = \text{augmentation ideal in } S(\mathfrak{g})^{G(\mathbb{C})}. \]

\[ \text{Ass}(V) \subset \text{Ass}(I_V) \subset \text{zeros of aug ideal in } S(\mathfrak{g})^{G(\mathbb{C})}. \]

**Theorem**

*If \(V\) is fin gen quasisimple module for reductive \(\mathfrak{g}\) (in particular, if \(V\) irreducible, then \(\text{Ass}(V)\) consists of nilpotent elts of \(\mathfrak{g}^*\).*
Howe’s wavefront set

...defined in Howe’s beautiful paper, which you should read. Defined for unitary \((\pi, \mathcal{H}_\pi)\) of Lie gp \(G\); def shows \(WF(\pi) \subset g_0^*\), closed cone preserved by coadjt action of \(G\). Definition involves wavefront sets of certain distributions \(T\) on \(G\) constructed using matrix coeffs of \(\pi\).

If \(\pi\) is quasisimple (automatic for irr unitary \(\pi\), by thm of Segal in Lec 5) then such \(T\) has \((\partial(z) - \chi_\pi(z))T = 0\).

Distribution on right above is smooth, so wavefront set is zero. Basic smoothness thm: applying diff op \(D\) can decrease wavefront set only by zeros of \(\sigma(D)\).

So \(WF(T) \subset\) zeros of \(\sigma(z)\), all \(z \in \mathfrak{z}(g)\) of pos deg:

\[WF(\pi) \subset\) zeros of augmentation ideal in \(S(g)^G(\mathbb{C})\).

Same proof: \(WF(\pi) \subset Ass(Ann(\mathcal{H}_\pi))\).

So \(WF(\pi)\) gives \(G\)-inv\(t\) subset of \(g_0^*\) sharing many props of \(Ass(V_\pi) \sim better\) classical limit than \(Ass(V_\pi)\).

But for reductive \(G\), \(WF(\pi), Ass(V_\pi)\) computable from each other (Schmid-Vilonen); so pick by preference.
Principal series revisited

Recall complete flag in $m$-diml vector space $E$ is

$$\mathcal{F} = \{0 = F_0 \subset F_1 \subset \cdots \subset F_m = E\}, \quad \text{dim } F_i = i.$$ 

Recall construction of principal series representations:

$$G = GL(n, k) \supset B = \text{upper triangular matrices}$$

$$X_n(k) = \text{complete flags in } k^n \cong G/B.$$ 

Fixing $n$ characters (group homomorphisms)

$$\xi_j : k^\times \rightarrow \mathbb{C}^\times$$

defines complex line bundle $L^\xi$;

$$V^\xi = \text{secs of } L^\xi \cong \{ f : G \rightarrow \mathbb{C} \mid f(gb) = \xi(b)^{-1}f(g) \ (b \in B) \},$$

$$\begin{pmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ 0 & b_{22} & \cdots & b_{2n} \\ \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & b_{nn} \end{pmatrix} = \xi_1(b_{11})\xi_2(b_{22})\cdots\xi_n(b_{nn}).$$

principal series rep of $GL(n, k)$ with param $\xi$.

Appropriate choice of topological vector space $V^\xi$ (continuous, smooth, $L^2\ldots$) depends on the problem.

$k = \mathbb{R}$: character $\xi$ is $(\nu, \epsilon) \in \mathbb{C} \times \mathbb{Z}/2\mathbb{Z}$, $t \mapsto |t|^\nu \text{ sgn}(t)^\epsilon$.
Principal series for $SL(2, \mathbb{R})$

Want to understand principal series repns for $(GL(2, \mathbb{R})$ restricted to) $SL(2, \mathbb{R})$. Helpful to use different picture

$$W^{\nu, \epsilon} = \{ f : (\mathbb{R}^2 - 0) \rightarrow \mathbb{C} \mid f(tx) = |t|^{-\nu} \text{sgn}(t)^{\epsilon} f(x) \},$$

functions on the plane homog of degree $-(\nu, \epsilon)$.

Exercise: $V^{(\nu_1, \nu_2)}(\epsilon_1, \epsilon_2)|_{SL(2, \mathbb{R})} \simeq W^{\nu_1 - \nu_2, \epsilon_1 - \epsilon_2}$.

Lie algs easier than Lie gps $\rightsquigarrow$ write $sl(2, \mathbb{R})$ action, basis

$$D = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad E = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad F = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix},$$

$$[D, E] = 2E, \quad [D, F] = -2F, \quad [E, F] = D.$$  

action on functions on $\mathbb{R}^2$ is by

$$D = -x_1 \frac{\partial}{\partial x_1} + x_2 \frac{\partial}{\partial x_2}, \quad E = -x_2 \frac{\partial}{\partial x_1}, \quad F = -x_1 \frac{\partial}{\partial x_2}.$$  

Now want to restrict to homogeneous functions...
Principal series for $\text{SL}(2, \mathbb{R})$ (continued)

Study homog fns on $\mathbb{R}^2 - 0$ by restr to $\{(\cos \theta, \sin \theta)\}$:

$W^{\nu, \epsilon} \simeq \{w: S^1 \to \mathbb{C} | w(-s) = (-1)^{\epsilon} w(s)\}$, $f(r, \theta) = r^{-\nu} w(\theta)$.

Compute Lie algebra action in polar coords using

$$
\frac{\partial}{\partial x_1} = -x_2 \frac{\partial}{\partial \theta} + x_1 \frac{\partial}{\partial r}, \quad \frac{\partial}{\partial x_2} = x_1 \frac{\partial}{\partial \theta} + x_2 \frac{\partial}{\partial r},
$$

$$
\frac{\partial}{\partial r} = -\nu, \quad x_1 = \cos \theta, \quad x_2 = \sin \theta.
$$

Plug into formulas on preceding slide: get

$$
\rho^\nu(D) = 2 \sin \theta \cos \theta \frac{\partial}{\partial \theta} + (- \cos^2 \theta + \sin^2 \theta) \nu,
$$

$$
\rho^\nu(E) = \sin^2 \theta \frac{\partial}{\partial \theta} + (- \cos \theta \sin \theta) \nu,
$$

$$
\rho^\nu(F) = - \cos^2 \theta \frac{\partial}{\partial \theta} + (- \cos \theta \sin \theta) \nu.
$$

Hard to make sense of. Clear: family of reps analytic (actually linear) in complex parameter $\nu$.

Big idea: see how properties change as function of $\nu$. 
A more suitable basis

Have family $\rho^{\nu,\epsilon}$ of reps of $SL(2, \mathbb{R})$ defined on functions on $S^1$ of homogeneity (or parity) $\epsilon$:

\[
\rho^{\nu}(D) = 2 \sin \theta \cos \theta \frac{\partial}{\partial \theta} + (-\cos^2 \theta + \sin^2 \theta) \nu,
\]
\[
\rho^{\nu}(E) = \sin^2 \theta \frac{\partial}{\partial \theta} + (-\cos \theta \sin \theta) \nu,
\]
\[
\rho^{\nu}(F) = -\cos^2 \theta \frac{\partial}{\partial \theta} + (-\cos \theta \sin \theta) \nu.
\]

Problem: $\{D, E, F\}$ adapted to wt vectors for diagonal Cartan subalgebra; rep $\rho^{\nu,\epsilon}$ has no such wt vectors.

But rotation matrix $E - F$ acts simply by $\partial/\partial \theta$.

Suggests new basis of the complexified Lie algebra:

\[
H = -i(E - F), \quad X = \frac{1}{2}(D + iE + iF), \quad Y = \frac{1}{2}(D - iE - iF).
\]

Same commutation relations as $D$, $E$, and $F$

\[
[H, X] = 2X, \quad [H, Y] = -2Y, \quad [X, Y] = H
\]

but complex conjugation is different: $\overline{H} = -H, \overline{X} = Y$.

\[
\rho^{\nu}(H) = \frac{1}{i} \frac{\partial}{\partial \theta}, \quad \rho^{\nu}(X) = \frac{e^{2i\theta}}{2i} \left( \frac{\partial}{\partial \theta} + i\nu \right), \quad \rho^{\nu}(Y) = \frac{-e^{-2i\theta}}{2i} \left( \frac{\partial}{\partial \theta} + i\nu \right).
\]
Matrices for principal series, bad news

Have family $\rho^{\nu,\epsilon}$ of reps of $SL(2, \mathbb{R})$ defined on functions on $S^1$ of homogeneity (or parity) $\epsilon$:

$$\rho^{\nu}(H) = \frac{1}{i} \frac{\partial}{\partial \theta}, \quad \rho^{\nu}(X) = \frac{e^{2i\theta}}{2i} \left( \frac{\partial}{\partial \theta} + i\nu \right), \quad \rho^{\nu}(Y) = \frac{-e^{-2i\theta}}{2i} \left( \frac{\partial}{\partial \theta} + i\nu \right).$$

These ops act simply on basis $w_m(\cos \theta, \sin \theta) = e^{im\theta}$:

$$\rho^{\nu}(H) w_m = mw_m,$$
$$\rho^{\nu}(X) w_m = \frac{1}{2} (m + \nu) w_{m+2},$$
$$\rho^{\nu}(Y) w_m = \frac{1}{2} (-m + \nu) w_{m-2}.$$

Suggests reasonable function space to consider:

$$W^{\nu,\epsilon,K} = \text{fns homog of deg } (\nu, \epsilon), \text{ finite under rotation}$$
$$= \text{span}(\{ w_m \mid m \equiv \epsilon \pmod{2} \}).$$

Space $W^{\nu,\epsilon,K}$ has beautiful rep of $g$: irr for most $\nu$, easy submods otherwise. Not preserved by rep of $G = SL(2, \mathbb{R})$:

$$\exp(A) \in G \leadsto \sum A^k/k! : A^k \text{ preserves } W^{\nu,\epsilon,K}, \text{ sum need not.}$$
Structure of principal series: good news

Original question was action of $G = SL(2, \mathbb{R})$ on

$$W^{\nu,\epsilon,\infty} = \{ f \in C^\infty(\mathbb{R}^2 - 0) \mid f \text{ homog of deg } -(\nu, \epsilon) \} :$$

what are the closed $G$-invt subspaces...?

Found nice subspace $W^{\nu,\epsilon,K}$, explicit basis, explicit action of Lie algebra $\rightsquigarrow$ easy to describe $g$–invt subspaces.

Theorem (Harish-Chandra tiny)

There is a one-to-one corr closed $G$-invt subspaces $S \subset W^{\nu,\epsilon,\infty}$ and $g$-invt subspaces $S^K \subset W^{\nu,\epsilon,K}$. Corr is $S \rightsquigarrow$ subspace of $K$-finite vectors, and $S^K \rightsquigarrow$ its closure:

$$S^K = \{ s \in S \mid \dim \text{span} (\rho^{\nu,\epsilon}(SO(2))s) < \infty \}, \quad S = \overline{S^K}.$$

Content of thm: closure carries $g$-invt to $G$-invt.

Why this isn’t obvious: $SO(2)$ acting by translation on $C^\infty(S^1)$. Lie alg acts by $\frac{d}{d\theta}$, so closed subspace

$$E = \{ f \in C^\infty(S^1) \mid f(\cos \theta, \sin \theta) = 0, \theta \in (-\pi/2, \pi/2) + 2\pi\mathbb{Z} \}$$

is preserved by $so(2)$; not preserved by rotation.

Reason: Taylor series for in $f \in E$ doesn’t converge to $f$. 
Same formalism, general $G$

Lesson of $SL(2, \mathbb{R})$ princ series: vecs finite under $SO(2)$ have nice/comprehensible/meaningful Lie algebra action.

General structure theory: $G = G(\mathbb{R})$ real pts of conn reductive complex algebraic group $\leadsto$ can embed $G \hookrightarrow GL(n, \mathbb{R})$, stable by transpose, $G/G_0$ finite.

Recall polar decomposition:

$$GL(n, \mathbb{R}) = O(n) \times (\text{pos def symmetric matrices})$$
$$= O(n) \times \exp(\text{symmetric matrices}).$$

Inherited by $G$ as Cartan decomposition for $G$:

$$K = O(n) \cap G, \quad s_0 = g_0 \cap (\text{symm mats}), \quad S = \exp(s_0)$$

$$G = K \times S = K \times \exp(s_0).$$

$(\rho, W)$ rep of $G$ on complete loc cvx top vec $W$;

$$W^K = \{ w \in W \mid \dim \text{span}(\rho(K)w) < \infty \},$$

$$W^\infty = \{ w \in W \mid G \to W, g \mapsto \rho(g)w \text{ smooth} \}.$$ 

**Definition.** The $(g, K)$-module of $W$ is $W^{K, \infty}$. It is a representation of the Lie algebra $g$ and of the group $K$. 
Category of \((\mathfrak{h}, L)\)-modules

Setting: \(\mathfrak{h} \supseteq \mathfrak{l}\) complex Lie algebras, \(L\) compact Lie group acting on \(\mathfrak{h}\) by Lie alg auts \(\text{Ad}\).

Definition

An \((\mathfrak{h}, L)\)-module is complex vector space \(W\) endowed with reps of \(\mathfrak{h}\) and of \(L\), subject to following conds.

1. Each \(w \in W\) belongs to fin-diml \(L\)-invt \(W_0\), such that action of \(L\) on \(W_0\) continuous (hence smooth).
2. Complexified differential of \(L\) action is \(\mathfrak{l}\) action.
3. For \(k \in L, Z \in \mathfrak{h}, w \in W,\)
   \[ k \cdot (Z \cdot (k^{-1} \cdot w)) = [\text{Ad}(k)(Z)] \cdot w. \]

Proposition

Passage to smooth \(K\)-finite vectors defines a functor

\[
\text{(reps of } G \text{ on complete loc cvx } W) \rightarrow (\mathfrak{g}, K)\text{-mods } W^{K, \infty}.
\]
Representations and $R$-modules

Rings and modules familiar and powerful $\rightsquigarrow$ try to make representation categories into module categories.

Category of reps of $\mathfrak{h} =$ category of $U(\mathfrak{h})$-modules.

Seek parallel for locally finite reps of compact $L$:

$$R(L) = \text{convolution alg of } C\text{-valued } L\text{-finite msres on } L$$

$$\simeq \sum_{(\mu, E_\mu) \in \hat{L}} \text{End}(E_\mu) \quad \text{(Peter-Weyl)}$$

$1 \notin R(L)$ if $L$ is infinite: convolution identity is delta function at $e \in L$; not $L$-finite.

$\alpha \subset \hat{L}$ finite $\rightsquigarrow 1_\alpha = \text{def} \sum_{\mu \in \alpha} \text{Id}_\mu$.

Elts $1_\alpha$ are *approximate identity* in $R(L)$: for all $r \in R(L)$ there is $\alpha(r)$ finite so $1_\beta \cdot r = r \cdot 1_\beta = r$ if $\beta \supset \alpha(r)$.

$R(L)$-module $M$ is *approximately unital* if for all $m \in M$ there is $\alpha(m)$ finite so $1_\beta \cdot m = m$ if $\beta \supset \alpha(m)$.

Loc fin reps of $L = \text{approx unital } R(L)$-modules.

If ring $R$ has approx ident $\{1_\alpha\}_{\alpha \in S}$, write $R\text{-mod}$ for category of approx unital $R$-modules.
Hecke algebras

Setting: $\mathfrak{h} \supset I$ complex Lie algebras, $L$ compact Lie group acting on $\mathfrak{h}$ by Lie alg auts $\text{Ad}$.

Definition

The Hecke algebra $R(\mathfrak{h}, L)$ is

$$R(\mathfrak{h}, L) = U(\mathfrak{h}) \otimes_{U(I)} R(L)$$

$$\simeq [\text{conv alg of } L\text{-finite } U(\mathfrak{h})\text{-valued msres on } L]/[U(I) \text{ action}]$$

$R(\mathfrak{h}, L)$ inherits approx identity from subalg $R(L)$.

Proposition

Category of $(\mathfrak{h}, L)$-modules is category $R(\mathfrak{h}, L)\text{-mod}$ of approx unital modules for Hecke algebra $R(\mathfrak{h}, L)$.

Exercise: repeat with $L$ cplx alg gp (not nec reductive).

Immediate corollary: category of $(\mathfrak{h}, L)$-modules has projective resolutions, so derived functors . . .

Lecture 7: use easy change-of-ring functors to construct $(\mathfrak{g}, K)$-modules.
Group reps and Lie algebra reps

$G$ real reductive alg $\supset K$ max cpt, $\mathcal{Z}(g) = \text{center of } U(g)$.

**Definition**

Rep $(\pi, V)$ of $G$ on complete loc cvx $V$ is *quasisimple* if $\pi^{\infty}(z) = \text{scalar}, \text{all } z \in \mathcal{Z}(g)$. Alg hom $\chi_{\pi} : \mathcal{Z}(g) \rightarrow \mathbb{C}$ is the *infinitesimal character of* $\pi$.

Make exactly same defn for $(g, K)$-modules.

**Theorem (Segal, Harish-Chandra)**

1. Any irr $(g, K)$-module is quasisimple.
2. Any irr unitary rep of $G$ is quasisimple.
3. Suppose $V$ quasisimple rep of $G$. Then $W \mapsto W^K,^{\infty}$ is bij $[\text{closed } W \subset V]$ and $[W^K,^{\infty} \subset V^K,^{\infty}]$.
4. Correspondence (irr quasisimple reps of $G$) $\sim \rightarrow$ (irr $(g, K)$-modules) is surjective. Fibers are infinitesimal equiv classes of irr quasisimple reps of $G$.

Non-quasisimple irr reps exist if $G'$ noncompact (Soergel), but are “pathological;” unrelated to harmonic analysis.

Idea of proof: $G/K \simeq s_0$, vector space. *Describe anything analytic on* $G$ *by Taylor exp along* $K$.  

**Note:** The content provided is a representative text sample extracted from a larger document. It reflects the major points covered in the specified sections, offering a synthesis of the core ideas without the extensive detail present in the original text.