Let P be a profinite group.

A - group with compatible P-action. $P \to \mathrm{Aut}(A)$

$$\begin{array}{c} 1 \\ \downarrow \\ A \\ \downarrow \\ P \ltimes A \\ \downarrow \uparrow \\ P \\ \downarrow \\ 1 \end{array}$$

$H^0(P, A) = A^P$. $\quad p(a)p(b) = p(ab)$

$Z^1(P, A) = \{$ continuous maps $P \to A$, $s \mapsto a_s$ such that $a_{st} = a_s \cdot s(a_t) \}$

Coboundary relation:

$a_s \sim a_s'$ if $b^{-1} a_s\, s(b) = a_s'$ for some $b \in A$.

Real forms: $H^1(\sigma, G)$ = elements x in $\sigma \ltimes G$ such that $x \in \sigma G$ with $x^2 = \mathrm{id}$ up to G-conjugacy

Gal action — real alg group

$x = \sigma \cdot g$

$x^2 = (\sigma \cdot g)(\sigma \cdot g) = \sigma(g) g$

Prop Let $\varphi: A \hookrightarrow B$ be an inclusion of groups respecting P-action
$(\varphi(p(a)) = p(\varphi(a)))$

$$0 \to H^0(P,A) \to H^0(P,B) \to (B/A)^P \xrightarrow{\delta} H^1(P,A) \xrightarrow{H^1(\varphi)} H^1(P,B)$$

is an exact sequence of pointed sets.

δ is defined by sending bA to $b^{-1}s(b)$.
The B^P-orbits of $(B/A)^P$ can be identified with $\operatorname{im} \delta$.

Exactness at $H^1(P,A)$: Suppose a_s is a cocycle for $Z^1(P,A)$ such that a_s maps to the trivial class in $H^1(P,B)$.

$a_s = b^{-1} s(b)$ for some $b \in B$.

$s(bA) = s(b)s(A) = b a_s\, s(A) = bA$

so bA is fixed, and a_s is in the image of δ.

Suppose $\delta(bA) = \delta(b'A)$.

$p(bA) = bA$
$p(b'A) = b'A$ for all $p \in P$.

$b^{-1}s(b) = x^{-1}(b')^{-1}s(b')s(x) \quad x \in A$

$\underbrace{b'xb^{-1}}_{\text{element of } B \text{ fixed by } P} = s(b'xb^{-1})$

$(b'xb^{-1})(bA) = b'A$

Coboundary condition

Example 1: $G(F)$-conjugacy classes of maximal tori.

Let F be a finite field, G connected reductive group over F.

$P = \mathrm{Gal}(\overline{F}/F) \simeq \hat{\mathbb{Z}}$

Carter's book: $G(F)$-conj classes of maxl tori $\longleftrightarrow$ Frob twisted conj classes in $W = N(T)/T$ for maxl torus T

Let T be a maxl torus defined over F, so T is P-stable, so $N(T)$ and $G/N(T)$ are defined over F.

Maxl tori defined over F

$$0 \to N(T)(F) \to G(F) \to \big[(G/N(T))(F)\big] \to H^1(P, N(T)) \to H^1(P, G)$$

$G(F)$-orbits in $G/N(T)(F) \longleftrightarrow$ kernel of $H^1(P, N(T)) \to H^1(P,G)$

FACT: Over F a finite field, with alg group C defined over F,

$$H^1(P, C) \simeq H^1(P, C/C_0)$$

$$\begin{array}{ccc} H^1(P, N(T)) & \longrightarrow & H^1(P, G) \\ \wr\!\! & & \| \ G \text{ is connected} \\ H^1(P, W) & & \{*\} \end{array}$$

$Z^1(P, W) = \{$ continuous maps $P \simeq \hat{\mathbb{Z}}$ to W $(s \mapsto w_s)$ such that $w_{st} = w_s\, s(w_t) \}$

P is topologically generated by the Frobenius (denoted Frob).

Each element in $Z^1(P, W)$ is determined by the image of Frob, so $Z^1(P,W) \xleftrightarrow{1-1} W$

Coboundary relations: $(s \mapsto w_s) \sim (s \mapsto w_s')$ if there is some $x \in W$, such that

$w_s' = x^{-1} w_s\, s(x)$ for all $s \in P$.

$\Longleftrightarrow \boxed{w'_{\mathrm{Frob}} = x^{-1} w_{\mathrm{Frob}}\, \mathrm{Frob}(x)}$

Example 2: Cartan subgroups of $GL(n, \mathbb{R})$.

Borovoi: Let G be a (Zariski) connected real reductive group.

Let H_f be a fundamental Cartan, so ($= Z(T)$, T maxl torus in K)

$$H_f(\mathbb{R}) \simeq (\mathbb{R}^\times)^a \times (\mathbb{C}^\times)^b \times (S^1)^c.$$

$H^1(P, H_f) = (S^1)^c_2$ = elements of order 2 in $(S^1)^c$

$$W_f = \left(\frac{N(T(\mathbb{C}))}{H_f(\mathbb{C})}\right)^P$$

W_f acts on $H^1(P, H_f)$ by sending

$w: (\sigma \mapsto z)$ to $(\sigma \mapsto n^{-1} z \bar{n})$

non trivial element in $P = \mathrm{Gal}(\mathbb{C}/\mathbb{R})$ — for n a representative of w in $N(T(\mathbb{C}))$.

$$H^1(P, G) \simeq H^1(P, H_f)/W_f$$