September 16, 2019: David Vogan (MIT), Involutive automorphisms.

In the first talk I said that the seminar would be about unitary representations of a real reductive algebraic group \( G(\mathbb{R}) \), and explained Harish-Chandra’s results relating those to Hermitian \((\mathfrak{g}(\mathbb{R}), K(\mathbb{R}))\)-modules (Harish-Chandra modules).

In this talk I’ll explain Elie Cartan’s proof that (isomorphism classes of) real reductive algebraic groups \( G(\mathbb{R}) \) are in one-to-one correspondence with (isomorphism classes of) involutive automorphisms \( \theta \) of a complex reductive algebraic \( G \). I’ll always write \( K = G^\theta \) for the (complex reductive algebraic) group of fixed points. For example, the real group \( GL(n, \mathbb{R}) \) corresponds to the involutive automorphism \( g \mapsto {}^t g^{-1} \) of \( GL(n, \mathbb{C}) \), with fixed point group \( K(\mathbb{C}) = O(n, \mathbb{C}) \).

I’ll explain how to reformulate the definition of Harish-Chandra modules in terms of \((G, \theta, K)\), so that it lives entirely in this complex algebraic world. Then I’ll look at the software (I hope putting it on zoom correctly this time!) to see how \( G \) and \( \theta \) and \( K \) are being described and computed internally.