On a conjecture of SR-V

Recall from bottom layer K-type talk: Cohomological Induction gives way to produce many irreducible and unitary reps, but knowing if a rep is unitary (or not unitary) can be very hard (unlike with parabolic induction).

TOOL: $\boxed{\text{Bottom layer K-types}}$ (basic form of unitary certificates?)

$$\mathcal{L}_S((\mathfrak{l}, L\cap K)\text{-rep}) = (\mathfrak{g}, K)\text{-rep} \Leftarrow \text{Unknown signature}$$

$$\cup \qquad\qquad \cup \text{ unitary inclusion}$$

$$\mathcal{L}_S^K((L\cap K)\text{ rep}) = K\text{-rep} \Leftarrow \text{Signature } \underline{\underline{\text{is}}} \text{ known}$$

HOPE: Want to find a way to produce reps in a way such that bottom K-types are sufficient (assuming induction from a smaller group).

One approach (Vogan's Green Book)

For K-type μ, can assign $P_a(\mu) \in \mathfrak{t}^*$ [A positive cone projection]

Let Λ_a be the set of all $P_a(\mu)$. Then let $B_a^{\lambda_a}(G)$ be set of $\delta \in \hat{K}$ s.t. $P_a(\mu) = \lambda_a$. Since $\mathfrak{g} = \mathfrak{t} \oplus [\mathfrak{t}, \mathfrak{k}] \oplus \mathfrak{p}$, there is a natural inclusion $\mathfrak{t}^* \subset \mathfrak{g}^*$ by ext by 0. Then let $G(\lambda_a)$ be centralizer of λ_a. Let $\Pi_a^{\lambda_a} = \{\text{irr admiss reps with lowest K-type in } B_a^{\lambda_a}(G)\}$. Then have the following:

Then ~~$B_a^{\lambda_a}(G)$ form~~ partition ~~of~~ admiss dual.

$\Pi_a^{\lambda_a}$

Thm • $\mathcal{L}_S^K$ gives bijection $B_a^{\lambda_a - \rho(u(\lambda_a))}(G(\lambda_a))$ onto $B_a^{\lambda_a}(G)$

• There is a natural bijection $\widehat{\Pi_a^{\lambda_a - \rho(u(\lambda_a))}(G(\lambda_a))} \longrightarrow \Pi_a^{\lambda_a}(G)$

defined as follows:

Take $Z_a \in \Pi_a^{\lambda_a - \rho(u(\lambda_a))}(G(\lambda_a))$.

Lowest $G(\lambda_a) \cap K$-types of Z_a are exactly those in $B_a^{\lambda_a - \rho(u(\lambda_a))}(G(\lambda_a))$

Lowest K-types of $\mathcal{L}_S(Z_a)$ given by $\mathcal{L}_S^K$(lowest K-types of Z_a).
(i.e. they are in $B_a^{\lambda_a}(G)$).

There is $\exists!$ subquotient of $\mathcal{L}_S(Z_a)$ with these K-types, so this gives map $Z_a \longrightarrow X$

$Z_a \in \Pi_a^{\lambda_a - \rho(u(\lambda_a))}(G_a)$, $X \in \Pi_a^{\lambda_a}(G)$.

• This bijection preserves Hermitian reps.

Great! So are we done? Unfortunately, no. This bijection does not give a correspondence of unitary reps.

[There are examples non-unitary $\longrightarrow$ unitary. One "explanation" for why this can happen is that bottom layer sends negative signature to zero.].

SR-V Idea: Change λ_a to some λ_a' such that $G(\lambda_a) \subset G(\lambda_a')$. Induction by stages gives analogue of above thm, but with new larger group. Induction is weaker, but since larger group has "bigger chance" of remembering unitarity, similarly "bigger chance" of unitary bijection.

Instead of $\mu \to P_a(\mu) = \lambda_a$, we have $\mu \to P_u(\mu) = \lambda_u$. Get sets $B_u^{\lambda_u}(G)$ like before (bigger sets), and get bijection

$$\Pi_h^{\lambda_u}(G(\lambda_u)) \longrightarrow \Pi_h^{\lambda_u}(G).$$

Signature on $G(\lambda_u) \cap K$-types $\longleftrightarrow$ Sig on K-type in $B_u^{\lambda_u}(G)$.

<u>Conjecture</u> (SR-V): This is a bijection of unitary reps.
This bijection respects the Fell topology (image for fixed λ_u is closed and open).

So, let's explain exactly how these λ_a are defined...

Other dir.

Fix h.w. $\mu \in i(\mathfrak{t}_0^c)^*$. [$\mathfrak{h} = \mathfrak{t}_0^c + \mathfrak{a}_0$ fundamental Cartan]

Pick $\Delta^+(\mathfrak{g}, \mathfrak{t}^c)$ so that $\mu + 2\rho_c$ dominant.

(Must have $\Delta^+(\mathfrak{g}, \mathfrak{t}^c) \supset \Delta^+(\mathfrak{k}, \mathfrak{t}^c)$)

Let $2\rho = \sum_{\gamma \in \Delta^+(\mathfrak{g}, \mathfrak{t}^c)} \gamma$ be sum of pos roots.

$P_u(\mu) =$ Projection to C of $\mu + 2\rho_c - 2\rho$

$$C = \{\xi \mid \langle \gamma, \xi \rangle \geq 0 \text{ for } \gamma \in \Delta^+(\mathfrak{g}, \mathfrak{t}^c)\}.$$

(This is all defined up to $R(G)$ - action).

Example: $G = SL(2)$. $K = SO(2)$. $\Delta(\mathfrak{g}, \mathfrak{t}) = \pm 2$

$\mu \in \hat{K} \cong \mathbb{Z}$. For $\mu \geq 0$, can pick $\Delta^+(\mathfrak{g}, \mathfrak{t}) = +2$.

Then get $P_u(\mu) =$ Projection of $\mu - 2$ onto $C = \{\xi \geq 0\}$.

Similarly, get $P_u(\mu)$ for $\mu \leq 0$ is Proj $\mu + 2$ onto $C = \{\xi \leq 0\}$.

Then