Unitarily Small Representations

LAST TIME: Discussed SR-V [1998]

- Groups representations of reductive group G according to "shifted projection map" P_u of K-type.

$$P_u(\mu) = (\mu + 2\rho_c) - 2\rho \text{ projected to positive cone corresponding to } \mu + 2\rho_c.$$

μ: h.w. of some irr of K

$$\Pi_h = \bigsqcup \underbrace{\Pi_h^{\lambda_u}}_{\substack{\text{all irr reps of } G \\ \text{with Hermitian form} \\ \text{with lowest } K\text{-type} \\ \text{projected to } \lambda_u.}}$$

Π_h: Hermitian reps of G

Conjecture: $\Pi_h^{\lambda_u}(G(\lambda_u)) \xrightarrow{\sim} \Pi_h^{\lambda_u}(G)$ preserves unitarity.

Today: Discuss $G(\lambda_u) = G$.

[μ satisfying this are called unitarily small K-types].

Let Z = connected component of center of G.
$Z_c = Z \cap K$.

Let G_s = derived subgroup of G_0.

$$T_S = T^c \cap G_S$$

Cartan in K

$$\mathfrak{g}_0 = \mathfrak{g}_{s,0} + \mathfrak{z}_0, \quad \mathfrak{t}_0^c = \mathfrak{t}_{s,0} \oplus \mathfrak{z}_{c,0}$$

Take $\mu \in i\mathfrak{t}_0^*$, then have

$$\mu = \underset{\substack{\cup\!\!\!|\\ i\mathfrak{z}_{s,0}^*}}{\mu_s} + \underset{\substack{\cup\!\!\!|\\ i\mathfrak{z}_{c,0}^*}}{\mu_z}$$

$$P_u(\mu) = P_u(\mu_s) + \mu_z$$

$\Rightarrow$ The μ_z is irrelevant, and μ is unitarily small iff μ_s is unitarily small.

Similarly, we can reduce to the simple case.

$$\mathfrak{g}_{s,0} = \mathfrak{g}_{s,0}^1 \oplus \mathfrak{g}_{s,0}^2 \oplus \cdots \oplus \mathfrak{g}_{s,0}^l$$

then $\mu_s = \mu_s^1 + \mu_s^2 + \cdots + \mu_s^l$

$$P_u(\mu_s) = P_u(\mu_s^1) + P_u(\mu_s^2) + \cdots + P_u(\mu_s^l).$$

so μ_s is unitarily small iff all of the μ_s^i are unitarily small.

Example. $G = U(2,1)$ $\quad K = U(2) \times U(1)$

$\mathfrak{z}$ = Diagonal $U(I) = Z_c$

$$\mu \in \hat{K} \longleftrightarrow (\underbrace{\mu_1^{(1)}, \mu_1^{(2)}}_{U(2)}, \underbrace{\mu_2^{(1)}}_{U(1)})$$

$$\mu_1^{(1)} \geq \mu_2^{(2)}$$

all integers.

$$\Delta(\mathfrak{g}, \mathfrak{t}^c) = \{ e_i - e_j \}_{i,j \in \{1,2,3\}}$$

$$2\rho_c = e_1 - e_2$$

2ρ depends on μ.

$$\mu + 2\rho_c = (\mu_1^{(1)} + 1, \mu_1^{(2)} - 1, \mu_2)$$

$$\begin{bmatrix} x & & \\ & y & \\ & & z \end{bmatrix}$$

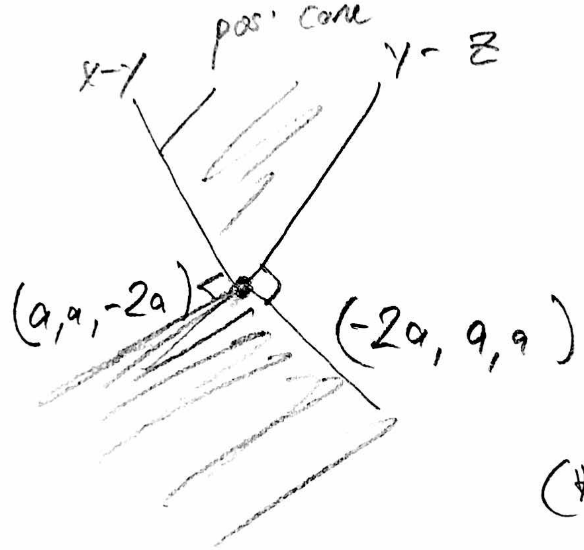
$x \geq y \geq z$

Cases: $\boxed{(1)\ \mu_1^{(1)} + 1 \geq \mu_1^{(2)} - 1 \geq \mu_2}$

(2) $\mu_1^{(1)} + 1 \geq \mu_2 \geq \mu_1^{(2)} - 1$

(3) $\mu_2 \geq \mu_1^{(1)} + 1 \geq \mu_1^{(2)} - 1$

We can assume $\mu_1^{(1)} + \mu_1^{(2)} + \mu_2 = 0$
(subtract $c(1,1,1)$ for some c).



$$\mu + 2\rho_c - 2\rho$$

$(2,0,-2)$

$$= (\mu_1^{(1)} - 1, \mu_1^{(2)} - 1, \mu_2 + 2)$$

$$(*) \quad \mu_1^{(1)} - 1 + \mu_1^{(2)} - 1 - 2(\mu_2 + 2) \geq 0$$

$$(**) \quad -2(\mu_1^{(1)} - 1) + \mu_1^{(2)} - 1 + \mu_2 + 2 \geq 0$$

Recall that $\mu_1^{(1)} + \mu_1^{(2)} + \mu_2 = 0$

$$(*) \quad -3\mu_2 - 6 \geq 0 \Leftrightarrow \mu_2 \leq 2$$

$$(**) \quad -3\mu_1^{(1)} + 3 \geq 0 \Leftrightarrow \mu_1 \leq 1$$

Examples of possible μs

$(1, 1, -2)$

$(-\frac{1}{3}, -\frac{1}{3}, \frac{2}{3}) \Leftrightarrow (0, 0, 1)$

Thm. The following are equivalent.

a) μ unitarily small.

b) $\lambda_u(\mu) = \mu_Z$

c) Let $\{\xi_1, \ldots, \xi_k\}$ be fundamental weights for our choice of $\Delta^+(\mathfrak{g}, \mathfrak{t}^c)$, and let $2\rho_n$ be the sum of the noncompact pos roots.

$$\langle \xi_i, \mu \rangle \leq \langle \xi_i, 2\rho_n \rangle \text{ for all } i$$

d) Let $\Delta^+(\mathfrak{g}, \mathfrak{t}^c)$ be any pos system containing $\Delta^+(\mathfrak{k}, \mathfrak{t})$, and let $2\rho_n$ be the sum of the noncompact pos roots.

$$\langle \xi_i, \mu \rangle \leq \langle \xi_i, 2\rho_n \rangle \text{ for all } i.$$

e) Let $\Delta^+(\mathfrak{g}, \mathfrak{t}^c)$ be a pos system containing $\Delta^+(\mathfrak{k}, \mathfrak{t}^c)$. Then

$$\mu = \mu_Z + \sum_{\beta \in \Delta^+(\mathfrak{p}, \mathfrak{t}^c)} c_\beta \beta \quad (c_\beta \in [0,1])$$

f) $$\mu = \mu_Z + \sum_{\beta \in \Delta(\mathfrak{p}, \mathfrak{t}^c)} b_\beta \beta \;(b_\beta \in [0,1])$$

Example for $G = U(2,1)$.

$\mu_2 \leq 2, \mu_1 \leq 1.$ $\quad \mu_1^{(1)} \geq 0$, $\mu_2^{(2)} \geq 0$ $\quad (e_1 - e_3)\mu_1^{(1)} + (e_2 - e_3)\mu_1^{(2)}$