Unipotently Small Representations

Last Time: Discussed SR-V [1992]
- Groups representations of reductive group $G$
  according to "shifted projection map" $P_u$ of $K$-type.

\[ P_u(\mu) = (\mu + 2p_c) - 2p \text{ projected to positive cone corresponding to } \mu + 2p_c. \]

h.w. of some
  h.w. of some
irr of $K$
  irreps of $K$

\[ \Pi^u_h = \bigcup \Pi^u_{h_k} \]

Hermitian reps
all irreps of $G$
with Hermitian form
of $G$
with lowest $K$-type
projected to $\Pi^u_k$.

\[ \Pi^u_h(G(\text{ad})) \cong \Pi^u_h(G) \]

preserves unitarity.

Today: Discuss $G(\text{ad}) = G$.

[$\mu$ satisfying this are called unitarily small $K$-types].

Let $Z = \text{connected component of center of } G$.

$Z_c = Z \cap K$.

Let $G_s = \text{derived subgroup of } G$. 
\[ T_s = T^c \cap G_s \]

Conten in \( K \)

\[ G_0 = G_{5,0} + Z_0 \quad \mathbf{A}^c_0 = \mathbf{A}_{5,0} + Z_{c,0} \]

Take \( \mu \in \text{Int}_0^* \) then have

\[ \mu = \mu_s + \mu_z \]

\[ \mathcal{A}^* \mathcal{H}_{5,0} \oplus \mathcal{A}^* \mathcal{H}_{c,0} \]

\[ P_{\mu}(\mu) = P_{\mu}(\mu_s) + \mu_z \]

\[ \Rightarrow \text{The } \mu_z \text{ is irrelevant and } \mu \text{ is } \]

unitarily small iff \( \mu_s \) is unitarily small.

Similarly, we can reduce to the simple case.

\[ G_{5,0} = G_{5,0}^1 \oplus G_{5,0}^2 \oplus \cdots \oplus G_{5,0}^c \]

Then \( \mu_s = \mu_s^1 + \mu_s^2 + \cdots + \mu_s^c \)

\[ P_{\mu}(\mu_s) = P_{\mu}(\mu_s^1) + P_{\mu}(\mu_s^2) + \cdots + P_{\mu}(\mu_s^c) \]

So \( \mu_s \) is unitarily small iff all of

the \( \mu_s^i \) are unitarily small.
Example. \( G = U(2, 1) \) \( K = U(2) \times U(1) \),
\[ \mathcal{Z} = \text{Diagonal} \quad U(2) = \mathbb{Z}_c \]
\[ \mu \in K \leftarrow \left( \mu_1, \mu_1^{(2)}, \mu_2^{(1)} \right) \]
\[ \mu_1 = \mu_2 \]
\[ \text{all integers}. \]

\[ \Delta(\mu, \mathcal{A}^c) = \{ e_i - e_j \} \]
\[ i, j \in \{ 1, 2, 3 \} \]

\[ 2pc = e_1 - e_2 \]

\( 2p \) depends on \( \mu \).

\[ \mu + 2pc = (\mu_1^{(1)} + 1, \mu_1^{(2)} - 1, \mu_2) \]

Cases:
1. \( \mu_1^{(1)} + 2\mu_2 - 1 > \mu_2 \)
2. \( \mu_1^{(1)} + 1 \geq \mu_2 \geq \mu_1^{(2)} - 1 \)
3. \( \mu_2 \geq \mu_1^{(1)} + 1 \geq \mu_1^{(2)} - 1 \)

\[ [x \ y \ z] \]

\[ x \geq y \geq z \]
We can assume \( \mu_1^{(1)} + \mu_1^{(2)} - \mu_2 = 0 \) 
(Subtract \( c \) for some \( c \)).

\[
\begin{align*}
\mu + 2pc &= 2p \\
(2, 0, -2) &= (\mu_1^{(1)} - 1, \mu_1^{(2)} - 1, \mu_2 + 2)
\end{align*}
\]

\( (\dagger) \quad \mu_1^{(1)} - 1 + \mu_1^{(2)} - 1 - 2(\mu_2 + 2) \geq 0 \)

\( (\dagger \dagger) \quad -2(\mu_1^{(1)} - 1) + \mu_1^{(2)} - 1 + \mu_2 + 2 \geq 0 \)

Recall that \( \mu_1^{(1)} + \mu_1^{(2)} + \mu_2 = 0 \)

\( (\dagger) \quad -3 \mu_2 - 6 \geq 0 \quad \Rightarrow \mu_2 \leq 2 \)

\( (\dagger \dagger) \quad -3 \mu_1^{(1)} + 3 \geq 0 \quad \Rightarrow \mu_1 \leq 1 \)

Examples of possible \( \mu \):

\[
(1, 1, -2)
\]

\[
(-\frac{1}{3}, -\frac{1}{3}, \frac{2}{3}) \iff (0, 0, 1)
\]
Thm. The following are equivalent:

a) $\mu$ unitarily small.

b) $\lambda u(\mu) = \mu z$

c) Let $\xi_1, \xi_2, \ldots, \xi_k, \xi$ be fundamental weights for our choice of $\Delta^+(\omega, \mathcal{A}^c)$, and let $2\rho_n$ be the sum of the noncompact positive roots.

$$\langle \xi_i, \mu \rangle \leq \langle \xi_i, 2\rho_n \rangle \text{ for all } i,$$

d) Let $\Delta^+(\omega, \mathcal{A}^c)$ be any pos system containing $\Delta^+(k, \mathcal{A})$, and let $2\rho_n$ be the sum of the noncompact positive roots.

$$\langle \xi_i, \mu \rangle \leq \langle \xi_i, 2\rho_n \rangle \text{ for all } i,$$

e) Let $\Delta^+(\omega, \mathcal{A}^c)$ be a pos system containing $\Delta^+(k, \mathcal{A}^c)$. Then

$$\mu = \mu z + \sum_{\beta \in \Delta(p, \mathcal{A}^c)} c_{\beta} \beta \quad \text{ (} c_{\beta} \in \mathbb{C}, \mathbb{I} \text{)}$$

f) $\mu = \mu z + \sum_{\beta \in \Delta(p, \mathcal{A}^c)} b_{\beta} \beta \quad \text{ (} b_{\beta} \in \mathbb{C}, \mathbb{I} \text{)}$

Example for $G = \mathcal{U}(2,1)$,

$$\mu_2 \leq 2, \quad \mu_1 \leq 1, \quad \mu_1^{(1)} \geq 0, \quad (e_1 - e_3) \mu_1^{(1)},$$

$$\mu_2^{(2)} > 0, \quad (e_2 - e_3) \mu_1^{(2)}.$$