What’s special about special?

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Outline

Introduction

Defining $WF(\pi)$

Listing nilpotent orbits

Structure of nilpotent orbits

Meaning of integral structure

Lusztig's definition of special

Special nilpotents and integral representations

Section titles are just getting longer. Glad that was the last one
What this talk is about

\( G(\mathbb{R}) \) real reductive algebraic group.

Example: \( Sp(2n, \mathbb{R}) \)

\((\pi, \mathcal{H}_\pi)\) irreducible (usually \( \infty \)-diml) rep of \( G(\mathbb{R}) \).

Example: \( \mathcal{H}_\pi = \) half-densities on \( \mathbb{R}P^{2n-1} \).

Study \( \pi \mapsto WF(\pi) \), a \textit{simple geometric invariant} of \( \pi \).

\( WF(\pi) \subset g(\mathbb{R})^* \), closed \( G(\mathbb{R}) \)-invt cone.

Ex: \( WF(\text{half-dens on } \mathbb{R}P^{2n-1}) = \text{rk} \leq 2 \) nilp sympl.

Ex: \( WF(\text{generic irr of } Sp(2n, \mathbb{R})) = \text{all} \) nilp sympl.

\( WF \) encodes \textit{interesting information} about \( \pi \).

Easy algebra: \( G(\mathbb{R}) \) has finite \# nilp orbits on \( g(\mathbb{R})^* \).

Easy soft analysis: \( WF(\pi) = \text{finite union of nilp orbits} \).

Deep result from Lusztig: \( \pi \) “integral” \( \implies \) \( WF(\pi) \) special.

PLAN: sketch defs, sketch Geck, Dong-Yang integral def of special, ask for \textit{direct proof} of Lusztig \( \implies \) above.
Howe’s wavefront set

$\pi$ nice irr rep of $G(\mathbb{R})$ on Hilbert space $\mathcal{H}_\pi$.

Read Howe’s beautiful Wave front sets of reps of Lie groups for def of $WF(\pi) \subset g(\mathbb{R})^*$: soft analysis.

Outline: trace class ops $T$ on $\mathcal{H}_\pi \rightsquigarrow$ “matrix coeff”
distributions $\pi_T$ on $G \rightsquigarrow WF(\pi_T) \subset T^*G$.

Big idea for controlling $WF(\pi)$:

$z \in \text{Cent } U(g) \rightsquigarrow \pi(z) = \text{scalar}$

$\rightsquigarrow$ differential equation $(z - \pi(z)) \cdot \pi_T = 0$

$\rightsquigarrow WF(\pi_T) \subset$ zeros of symbol of $z$.

Symbols of $z \in \text{Cent } U(g)$ are homog polys $p \in S(g)^G$.

Real nilpotent cone (where $WF(\pi)$ must live!) is

$N^*_R = \{ \lambda \in g(\mathbb{R})^* \mid p(\lambda) = 0 \ (p \in S(g)^G \text{ homogeneous}) \}$.

$N^*_R / G(\mathbb{R})$ finite $\implies$ $WF(\pi) = \text{finite } \# \ G(\mathbb{R})$ orbs.
Calculating nilpotent orbits

\(WF(\pi)\) is elementary, uncomplicated invariant of \(\pi\).

Zeroth problem: describe \(G(\mathbb{R})\) orbits \(O_\mathbb{R}\) on \(g(\mathbb{R})^*\).

\(T_s(\mathbb{R}) \subset G(\mathbb{R})\) Iwasawa (most split) max torus.

\[
X_{*,h}(T_s) = \text{Hom}_{\text{alg}}(\mathbb{R}^\times, T_s(\mathbb{R})) \subset \text{Hom}_{\text{alg}}(\mathbb{C}^\times, T_s) = X_*(T_s).
\]

\(d \in X_{*,h}(T_s) \mapsto \text{Lie algebra } \mathbb{Z}\text{-grading}
\[
g(\mathbb{R}) = \sum_{n \in \mathbb{Z}} g(\mathbb{R})_d(n), \quad ts(\mathbb{R}) \subset g(\mathbb{R})_d(0).
\]

Levi \(G(\mathbb{R})^d\) has open orbits on each \(g(\mathbb{R})_d^*(n)\) \((n \neq 0)\)

Thm (Jacobson-Morozov) \(O_\mathbb{R} \subset N_\mathbb{R}^* \mapsto d \in X_{*,h}(T_s)\) so

\(O_\mathbb{R}\) meets \(g(\mathbb{R})_d^*(2)\) in open, \(d \in [g(\mathbb{R})_d(2), g(\mathbb{R})_d(-2)]\).

This defines a finite-to-one map

\[
N_\mathbb{R}^*/G(\mathbb{R}) \mapsto X_{*,h}(T_s)/W_s(\mathbb{R}) \simeq \text{dom cowts } X_+^{*,h}(T_s).
\]

Fiber over \(d \leftrightarrow\) open orbits of \(G(\mathbb{R})^d\) on \(g(\mathbb{R})_d^*(2)\)

Dominant coweight \(d\) called the Dynkin diagram of \(O_\mathbb{R}\).
Structure of orbits

Nilp orb $\mathcal{O}_\mathbb{R} \rightsquigarrow \text{dom } d \in \text{Hom}_{\text{alg}}(\mathbb{R}^\times, T_s(\mathbb{R}))$,

$\mathcal{O}_\mathbb{R}$ meets $g(\mathbb{R})^*_d(2)$ in open, $d \in [g(\mathbb{R})_d(2), g(\mathbb{R})_d(-2)]$.

$$g(\mathbb{R})_d(n) = \{X \in g(\mathbb{R}) \mid [d, X] = nX\}$$

$$g(\mathbb{R})^*_d(n) = [g(\mathbb{R})_d(-n)]^*.$$ 

If $\alpha \in \mathbb{R}^+$ simple, then $\alpha(d) = 0$ or $1$ or $2$.

Partition simple roots $\Pi$ as $\Pi_d(0) \cup \Pi_d(1) \cup \Pi_d(2)$.

$G^d = \text{Levi subgp } \leftrightarrow \Pi_d(0)$

$g_d(-1) = \text{sum of } G^d \text{ irrs, hwts } - \alpha \in \Pi_d(1)$

$P = G^d U, \quad u = \sum_{n>0} g_d(n).$

Fix $\lambda \in \mathcal{O}_\mathbb{R} \cap g(\mathbb{R})^*_d(2)$. Then $G^\lambda \subset P$, and

$$G^\lambda = [G^d]^\lambda \cdot U^\lambda \quad \text{(Levi decomp)}$$

$$u^\lambda = \sum_{n>0} g_\lambda^d(n),$$

all decompositions defined over $\mathbb{R}$. 
Symplectic structure on orbits

Nilp orb $\mathcal{O}_R \leadsto \text{dom } d \in \text{Hom}_{\text{alg}}(\mathbb{R}^\times, T_s(\mathbb{R}))$,

$\mathcal{O}_R$ meets $g(\mathbb{R})^*_d(2)$ in open, $d \in [g(\mathbb{R})_d(2), g(\mathbb{R})_d(-2)]$.

$\lambda \in \mathcal{O}_R \cap g(\mathbb{R})^*_d(2)$. Then $G^\lambda \subset P$, and

$G^\lambda = [G^d]^\lambda \cdot U^\lambda$ (Levi decom)

$u^\lambda = \sum_{n>0} g^\lambda_d(n)$

$T_{eG^\lambda}(G \cdot \lambda) = g/g^\lambda$

$= g_d(-1) + \sum_{m\geq0} \left[g_d(-m-2) + g_d(m)/g_d(m)^\lambda\right]$.

$\mathcal{O}_R$ is a symplectic manifold: nondegenerate form

$\omega_\lambda : g(\mathbb{R})/g(\mathbb{R})^\lambda \times g(\mathbb{R})/g(\mathbb{R})^\lambda \rightarrow \mathbb{R}$

$\omega_\lambda(X, Y) = \lambda([X, Y])$

$[g_d(-m-2)]^* \simeq_{\omega_\lambda} g_d(m)/g_d(m)^\lambda \quad (m \geq 0)$

$\omega_\lambda$ nondegenerate on $g_d(-1)$.

$\omega_\lambda$ needed to relate $\mathcal{O}_R$ to representation theory.

Geck conj: $\mathcal{O}_R$ special $\iff$ $\omega_\lambda$ integral (to be explained).
Integral structures on $\mathfrak{g}$

Integral structure on $N$-diml Lie algebra $\mathfrak{g}$ over char 0 field $k$ is free rank $N$ lattice $\mathfrak{g}_{\mathbb{Z}} \subset \mathfrak{g}$ subject to

$$\mathfrak{g} = \mathfrak{g}_{\mathbb{Z}} \otimes_{\mathbb{Z}} k, \quad [\mathfrak{g}_{\mathbb{Z}}, \mathfrak{g}_{\mathbb{Z}}] \subset \mathfrak{g}_{\mathbb{Z}}.$$

Equivalent: basis $\{X_1, \ldots, X_N\}$ subject to

$$[X_i, X_j] = \sum_k c_{ij}^k X_k, \quad c_{ij}^k \in \mathbb{Z}.$$

Example: $\mathfrak{g} = \mathfrak{sl}(2)$, basis (this one we’ll generalize)

$$H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad X = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad Y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix},$$

$$[H, X] = 2X, \quad [H, Y] = -2Y, \quad [X, Y] = H.$$

Example: $\mathfrak{g} = \mathfrak{so}(3)$, basis (but this is worth more study!)

$$U = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad V = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}, \quad W = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix},$$

Chevalley integral structure

\(g \supset b \supset t\) cplx reduc; roots \(\Delta(g, t) \subset t^*\), coroots \(\Delta^\vee(g, t) \subset t\).

Integral structure is called **split** if

1. Have integral basis = basis \(\{X_1, \ldots, X_\ell\}\) of \(t\), root vectors \(X_\alpha\) for each root; and
2. \([X_\alpha, X_{-\alpha}]\) is equal to the coroot \(H_\alpha = \alpha^\vee\).

**Chevalley**: in a split integral structure, set of root vecs up to sign \(\{\pm X_\alpha\}\) is determined up to \(\text{Ad}(T)\), so should be thought of as **unique**.

Still in a split integral structure,

\[\mathbb{Z}\Delta^\vee \subset t_{\mathbb{Z}} \subset \{t \in t \mid \alpha(t) \in \mathbb{Z} \quad (\alpha \in \Delta)\};\]

and any such lattice \(t_{\mathbb{Z}}\) is allowed.

These \(t_{\mathbb{Z}}\) are the \(X_\ast(T)\) root data for alg \(G\), \(\text{Lie}(G) = g\).

If \(g\) semisimple, split integral structure (unique up to \(\text{Ad}(T)\)) with \(t_{\mathbb{Z}} = \mathbb{Z}\Delta^\vee\) is the **Chevalley integral structure**.
Integral linear functionals

\[ \text{split int str } g_Z \subset g \leadsto g_Z^* = \text{def } \text{Hom}_\mathbb{Z}(g_Z, \mathbb{Z}) \subset g^*. \]

\( O_R \) weakly integral if \( O_R \cap g_Z^* \neq \emptyset \); includes nilpotent.

Precisely: elt \( d \in \text{coroot lattice} \subset t_\mathbb{Z}; O_R \) has element \( \lambda \in g_{d,\mathbb{Z}}^*(2): \lambda(X_\alpha) = c_\alpha \in \mathbb{Z} \ (\alpha \in \Delta, \alpha(d) = 2). \)

Symplectic form \( \omega_\lambda \) defines

\[ \omega_{\lambda,\mathbb{Z}}: g_Z/g_Z^\lambda \leftrightarrow [g_Z/g_Z^\lambda]^*. \]

Nondegen/\( \mathbb{R} \) implies \( \text{im}(\omega_{\lambda,\mathbb{Z}}) \) has finite index \( N_\lambda. \)

Weights of \( d \) decomposition factors \( \omega_{\lambda,\mathbb{Z}} \) as sum of

\[ \omega_{\lambda,\mathbb{Z}}(m): g_{d,\mathbb{Z}}(m)/g_{d,\mathbb{Z}}(m)^{\lambda} \leftrightarrow [g_{d,\mathbb{Z}}(-m-2)]^* \quad (m \geq 0), \]

\[ \omega_{\lambda,\mathbb{Z}}(-1): g_{d,\mathbb{Z}}(-1) \leftrightarrow g_{d,\mathbb{Z}}(-1)^*. \]

Each of these has finite index \( N_\lambda(m) \) in its image, and

\[ N_\lambda = N_\lambda(-1) \cdot \prod_{m \geq 0} N_\lambda(m). \]

\( \lambda \) is strongly integral if \( N_\lambda = 1 \); that is, if \( \omega_{\lambda,\mathbb{Z}} \) is nondeg/\( \mathbb{Z} \).

\( \lambda \) is Geck integral if \( N_\lambda(-1) = 1 \); that is, if \( \omega_{\lambda,\mathbb{Z}}(-1) \) is nondeg/\( \mathbb{Z} \).
Lusztig’s notion of special in $\hat{W}$

$W$ Weyl grp for $G \hookrightarrow$ Chevalley group $G(\mathbb{F}_q) \supset B(\mathbb{F}_q)$.

Natural bij $\sigma \leftrightarrow \pi_q(\sigma)$ between irrs $\sigma \in \hat{W}$ and irrs $\pi_q(\sigma)$ of $G(\mathbb{F}_q)$ appearing in fns on $G(\mathbb{F}_q)/B(\mathbb{F}_q)$.

generic degree $\tilde{P}_\sigma(q) = \text{def dim} \pi_q(\sigma)$: poly in $q$, $\mathbb{Q}$-coeffs.

Cpt mfld $X = G(\mathbb{C})/B(\mathbb{C})$: cohom only even degs.

$W$ acts naturally on $H^*(X)$. $\hookrightarrow$ regular rep of $W$.

Can therefore define fake degree

$$P_\sigma(q) = \sum_{i=0}^{r} (\text{mult of } \sigma \text{ in } H^{2i}(X))q^i$$
	poly in $q$, nonneg integer coeffs summing to $\text{dim } \sigma$.

$G = GL(n), \quad W = S_n$: $\tilde{P}_\sigma = P_\sigma$.

Define $\tilde{a}_\sigma = \text{least } q^a \text{ in } \tilde{P}_\sigma$, $a_\sigma = \text{least } q^a \text{ in } P_\sigma$.

Lusztig 1979: $\tilde{a}_\sigma \leq a_\sigma$; say $\sigma$ is special if $\tilde{a}_\sigma = a_\sigma$. 
Lusztig’s notion of special for nilpotent orbits

Springer (1978) defined inclusion $j$

$$j: \text{nilpotent orbits in } g^* \hookrightarrow \hat{W}, \quad O \mapsto j(O).$$

Easy: $\dim(O) = 2r - 2a_j(O)$ ($r = \#\text{pos roots}$).

Springer (1978) also defined surjection $p$ ($p \circ j = id$)

$$p: \hat{W} \twoheadrightarrow \text{nilpotent orbits in } g^*, \quad \sigma \mapsto p(\sigma).$$

Easy: $\dim(p(\sigma)) \geq 2r - 2a_\sigma$, equality iff $j \circ p(\sigma)) = \sigma$.

KL theory partitions $\hat{W}$ in families (two-sided cells).

Theorem (Lusztig)

1. Each family $\mathcal{F} \subset \hat{W}$ has unique special rep $\sigma_s(\mathcal{F})$.
2. Function $\tilde{a}_\sigma$ is constant on each family.
3. Function $a_\sigma$ has unique minimum on $\mathcal{F}$, at $\sigma_s(\mathcal{F})$.
4. $\sigma_s(\mathcal{F})$ is $j(O(\mathcal{F}))$, special nilpotent orbit.
Geck conj/Dong-Yang thm on special nilps

\[ G \supset B \supset T, \mathcal{O} \subset \mathfrak{g}^* \rightsquigarrow \text{Jacobson-Morozov dom } d \in X_*(T): \]
\[ d \in [\mathfrak{g}_d(2), \mathfrak{g}_d(-2)], \quad \mathcal{O} \cap \mathfrak{g}_d^*(2) \text{ open in } \mathfrak{g}_d^*(2) \]
\[ \rightsquigarrow \omega_\lambda \text{ symplectic on } \mathfrak{g}/\mathfrak{g}^\lambda, \quad \omega_\lambda(-1) \text{ on } \mathfrak{g}_d(-1) \subset \mathfrak{g}/\mathfrak{g}^\lambda. \]

Fix also split int str \( \mathfrak{g}_Z \subset \mathfrak{g} \rightsquigarrow \mathfrak{g}_Z^* \subset \mathfrak{g}^*. \)

May choose representative \( \lambda_Z \in \mathcal{O} \cap \mathfrak{g}_d^*(2). \)

Conj (Geck 2018) \( \mathcal{O} \) special iff \( \exists \lambda_Z \) so \( \omega_{\lambda_Z}(-1) \) nondeg/\( \mathbb{Z} \).

Proved by Geck (types \( \text{EFG} \)), Dong-Yang (2019) (types \( \text{ABCD} \)).

Proof is case-by-case using enumeration of special nilps.

Recall that hypothesis \textbf{Geck integral} in Geck conjecture is weaker than natural hypothesis \textbf{strongly integral}.

Hope: Geck integral \textbf{equivalent} to strongly integral.
Lusztig thm on special nilps

**Theorem** (Lusztig) Suppose \( \pi \) irr rep of real reductive \( G(\mathbb{R}) \) of integral infl char. Then there is a special \( \mathcal{O} \subset g^* \) so that \( WF(\pi) \) is closure of some real forms \( \mathcal{O}^i_R \) of \( \mathcal{O} \).

Proof is by KL theory, properties of families in \( \hat{W} \).

**Hope** (point of talk): there is a conceptual path

\[ \pi \text{ integral infl char} \rightsquigarrow WF(\pi) \text{ strongly integral.} \]

Such a path could give a conceptual proof

\[ (\mathcal{O} \text{ special}) \implies (\mathcal{O} \text{ str int}) \implies (\mathcal{O} \text{ Geck int}).\]

which is half of Geck’s conjecture.

**Path to Hope**: \( \exists \) nice \( \mathbb{Z} \)-forms of reps with int infl char.

I like this question. Can find in *Green Monster* (Vogan 1981) \( \mathbb{Z} \)-forms for \( SL(2, \mathbb{R}) \) reps in block of finite-diml.

First easy exercise: other blocks of int infl char. for \( SL(2, \mathbb{R}) \)?
Thank you!