February 19: David Vogan (MIT), Bruhat order on representations of $K$ II: nonunitarity certificates.

Last week I sketched a construction of a partial (pre)order on the set $\hat{K}$ of irreducible representations of the complexified maximal compact subgroup of a real reductive algebraic group. A central property of this order is that $\mu \leq \mu'$ if $\mu'$ appears in a standard $(\mathfrak{g}, K)$-module with lowest $K$-type $\mu$. For this week I will just define the Bruhat order to be the transitive closure of this relation; the geometric relation I considered last week at least includes all these relations, and probably they are exactly the same.

In order to classify unitary $(\mathfrak{g}, K)$-modules, it is important for each $\mu$ to find a small finite set of nonunitary certificates $\{\mu_1, \ldots, \mu_m\}$. These are $K$-representations $\mu_i \geq \mu$ with the property that if $J$ is an irreducible nonunitary Hermitian representation of lowest $K$-type $\mu$, then the form must be indefinite on some pair of $K$-types $(\mu, \mu_i)$. (Another way to say this is that if the form is positive on $\mu$, then it must fail to be positive on some $\mu_i$.)

For $G = GL(n, \mathbb{C})$ (regarded as a real group), $K = GL(n, \mathbb{C})$, and $\mu$ the trivial representation of $K$, the set of $[n/2]$ $K$-types

$$\mu_i = \text{irr of highest weight } (1, \ldots, 1, 0, \ldots, 0, -1, \ldots, -1)$$

is a set of nonunitarity certificates for $\mu$. (Here there are $i$ 1s and $-1$s, and $n - 2i$ 0s.) This statement is the main step in the classification of the unitary dual of $GL(n, \mathbb{C})$.

Just as teaser about Why This Is (related to) Interesting Mathematics, I will mention that if $G$ is a split group over a $p$-adic field and $K = G(\mathcal{O})$ the usual maximal compact, then a set of nonunitary certificates for the trivial representation of $K$ is indexed by the nontrivial representations of the Weyl group. This result of Barbasch and Moy was used by Barbasch and Ciubotaru to classify the spherical unitary representations of $G$.

I will outline an idea for finding (well, for guessing) a set of nonunitarity certificates for some $\mu$ and general $G$. The idea has some connection to the geometry from last week.