
This is the not-really-annual Free Pizza for Graduate Students Lie groups seminar. There won’t be any mathematics interesting to the usual faculty suspects.

One of the most basic results about a semisimple Lie algebra \( g \) is the Weyl character formula. One way to think about it (as explained in Humphreys’ book) is this. Start with a “triangular decomposition” \( g = n^- + h + n \). Here \( h \) is a Cartan subalgebra of \( g \) and \( b = h + n \) is a Borel subalgebra of \( g \).

Now one can attach to each linear functional \( \gamma \in h^* \) two modules for the Lie algebra \( g \). The most interesting is the “irreducible highest weight module” \( L(\gamma) \). The easiest is the “Verma module” \( M(\gamma) \). Each of these modules decomposes under \( h \) into a direct sum of finite-dimensional weight spaces:

\[
L(\gamma) = \sum_{\mu \in h^*} L(\gamma)(\mu),
\]

and similarly for \( M(\gamma) \). The fundamental problem of character theory is to compute \( \dim(L(\gamma)(\mu)) \) for every \( \gamma \) and \( \mu \). Here is a way to approach that problem.

The weight space dimensions for Verma modules are fairly accessible: there is an easy-to-compute integer-valued function \( P \) on \( h^* \) (the Kostant partition function) so that \( \dim M(\gamma)(\mu) = P(\mu - \gamma) \). As is explained in Humphreys, every \( L(\gamma) \) can be written as a finite integer combination of various Verma modules \( M(\gamma') \):

\[
L(\gamma) = \sum_{\gamma' \in h^*} c(\gamma', \gamma)M(\gamma').
\]

Now we can write

\[
\dim L(\gamma)(\mu) = \sum_{\gamma' \in h^*} c(\gamma', \gamma)P(\mu - \gamma').
\]

So we know these dimensions as soon as we know the integers \( c(\gamma', \gamma) \).

I’ll explain what these integers have to do with Lie algebra cohomology; what this formalism has to do with the Weyl character formula; and finally how Kazhdan-Lusztig theory lets you compute the integers \( c(\gamma', \gamma) \).