Problem Set 7

Due Monday, April 8 in class.

This problem set is about continued fractions. To fix the notation, I’ll write here a little of what’s written in the text. The starting point is two integers

\[ u_0, u_1, \quad u_1 \geq 1. \]

The algorithm for computing the continued fraction expansion is very much like the Euclidean algorithm: repeated division with remainder

\[
\begin{align*}
  u_0 &= u_1a_0 + u_2, \quad (0 \leq u_2 < u_1) \\
  u_1 &= u_2a_1 + u_3, \quad (0 \leq u_3 < u_2) \\
  & \vdots \\
  u_{n-1} &= u_na_{n-1} + u_{n+1} \quad (0 \leq u_{n+1} < u_n) \\
  u_n &= u_{n+1}a_n.
\end{align*}
\]

Then

\[
\frac{u_0}{u_1} = a_0 + \frac{1}{a_1 + \frac{1}{a_2 + \frac{1}{\ddots + \frac{1}{a_n}}}} = \frac{P_j}{Q_j} - \frac{1}{\frac{P_{j-1}}{Q_{j-1}}},
\]

\[ \text{def} \langle a_0, \ldots, a_n \rangle. \]

1a. The text says that you should start with a fraction \( u_0/u_1 \) in lowest terms; that is, with the property that \( u_0 \) and \( u_1 \) have no common factor. If you do that, what is the value of \( u_{n+1} \)?

1b. Explain what happens in the algorithm above if you start with a fraction \( u_0/u_1 \) that is \textbf{not} in lowest terms.

2. Define

\[
A_j = \begin{pmatrix} a_j & 1 \\ 1 & 0 \end{pmatrix},
\]

and define

\[
\begin{pmatrix} P_j & P_{j-1} \\ Q_j & Q_{j-1} \end{pmatrix} = A_0A_1 \cdots A_j. \quad (n \geq j \geq 0)
\]

Prove that for all \( 0 \leq j \leq n \)

\[
\langle a_0, \ldots, a_j \rangle = P_j/Q_j,
\]

\[
P_jQ_{j-1} - P_{j-1}Q_j = (-1)^{j+1},
\]

\[
Q_0 = 1, \quad Q_j \geq Q_{j-1},
\]

with equality only if \( j = 1 \) and \( a_1 = 1 \). Finally, for \( 1 \leq j \leq n \),

\[
\frac{P_j}{Q_j} - \frac{P_{j-1}}{Q_{j-1}} = \frac{(-1)^{j+1}}{Q_jQ_{j-1}}.
\]
3. If $x > 0$ is any real number, define

$$\langle a_0, \ldots, a_n, x \rangle = \text{def} = a_0 + \frac{1}{a_1 + \frac{1}{a_2 + \frac{1}{\ddots + \frac{1}{a_n + \frac{1}{x}}}}}$$

(If $x$ is a positive integer, this is consistent with our notation for continued fractions.) Using the notation of Problem 2, prove that

$$\langle a_0, \ldots, a_n, x \rangle = \frac{P_nx + P_{n-1}}{Q_nx + Q_{n-1}}.$$  

4. Find an explicit formula (something like $4 - 2\sqrt{3}$) for the periodic continued fraction

$$\langle 1, 2, 3, 1, 2, 3, 1, 2, 3, \ldots \rangle = \langle 1, 2, 3 \rangle.$$  

(Hint: if you use the previous problems, you can make most of the arithmetic into multiplying some $2 \times 2$ matrices.)