1. Representations of compact Lie groups

This is to write down some facts which are needed to do the homework. The main point is Proposition 1.4.

Setting is that we have a compact connected Lie group with a choice of maximal torus

\[(1a)\quad G \supset T.\]

Attached to the torus are two lattices

\[(1b)\quad X^*(T) = \text{Hom}(T,U(1)) = \text{character lattice},
X_*(T) = \text{Hom}(U(1),T) = \text{cocharacter lattice},\]

in duality by a pairing \((\cdot,\cdot)\). Write

\[(1c)\quad g_0 = \text{Lie}(G), \quad g = g_0 \otimes \mathbb{R} \mathbb{C}
\]

for the Lie algebra and its complexification; similar notation is used for other groups. The roots are defined by the weight decomposition of \(T\) in the adjoint representation

\[(1d)\quad g = \mathfrak{t} \oplus \sum_{\alpha \in R} g(\alpha);\]

then

\[(1e)\quad R = R(G,T) \subset X^*(T) \setminus \{0\}, \quad R^\vee = R^\vee(G,T) \subset X_*(T) \setminus \{0\},\]

where I have not recalled the definition of the coroots \(R^\vee\).

It’s often useful to have the rational vector spaces

\[(1f)\quad X^*(\mathbb{Q}) = \text{def} \ X^* \otimes_\mathbb{Z} \mathbb{Q}, \quad X_*(\mathbb{Q}) = \text{def} \ X_* \otimes_\mathbb{Z} \mathbb{Q};\]

these are dual vector spaces by the rational extension of \((\cdot,\cdot)\). The following definitions can be made for \(X^*\) or for \(X_*(\mathbb{Q})\). An element \(\xi \in X_*(\mathbb{Q})\) is \textit{regular} if and only if

\[(1g)\quad \langle \alpha, \xi \rangle \neq 0 \quad (\alpha \in R);\]

otherwise \(\xi\) is \textit{singular}. To each regular element \(\xi\) one can attach a \textit{system of positive roots}

\[(1h)\quad R^+(\xi) = \{ \beta \in R \mid \langle \beta, \xi \rangle > 0 \}.\]

We fix henceforth a system of positive roots, called just \(R^+\). Obviously \(R\) is the disjoint union of the positive and negative roots. A positive root is called \textit{simple} if it is not the sum of two other positive roots. Write

\[(1i)\quad \Pi = \{ \alpha \in R^+ \mid \alpha \text{ is simple} \}\]

for the set of simple roots.
Proposition 1.1. Every positive root can be written uniquely as a sum with nonnegative integer coefficients of simple roots. The simple roots are a basis for the subspace of $X^*(\mathbb{Q})$ spanned by the roots, and a basis for the sublattice of $X^*$ generated by the roots.

A weight $\mu \in X^*(\mathbb{Q})$ is called dominant if
\begin{equation}
\langle \mu, \alpha^\vee \rangle \geq 0 \quad (\alpha \in R^+);
\end{equation}
it is enough to check this positivity for $\alpha \in \Pi$. In exactly the same way we can define dominant coweights in $X_*(\mathbb{Q})$.

Proposition 1.2. Every orbit of the Weyl group on $X^*(\mathbb{Q})$ contains exactly one dominant weight.

Definition 1.3. Suppose $\mu$ and $\lambda$ belong to $X^*$. We write
\[ \mu \preceq \lambda \]
if $\lambda$ is equal to $\mu$ plus a nonnegative integer combination of positive roots.

Here is one description of the weights of an irreducible representation.

Proposition 1.4. Suppose $\xi \in X^*$ is dominant, and $\pi(\xi)$ is the irreducible representation of $G$ of highest weight $\xi$. Then the dominant weight $\mu \in X^*$ is a weight of $\pi(\xi)$ if and only if $\mu \preceq \xi$.

If $\tau \in X^*$ is arbitrary, then $\tau$ is a weight of $\pi(\xi)$ if and only if the dominant conjugate $w\tau$ satisfies $w\tau \preceq \xi$. In this case it is also true that $\tau \preceq \xi$.

One has to be careful with quantifiers: if $\tau$ is not dominant, it can happen that $\tau \preceq \xi$, but nevertheless $\tau$ is not a weight of $\pi(\xi)$.

Here is another description of the weights.

Proposition 1.5. Suppose $\xi \in X^*$ is dominant, and $\pi(\xi)$ is the irreducible representation of $G$ of highest weight $\xi$. Then the set of weights of $\pi(\xi)$ is equal to the intersection of
\[ \text{convex hull of } W \cdot \xi \subset X^*(\mathbb{Q}) \]
with the lattice coset
\[ \xi + \mathbb{Z}R \]
of translates of $\xi$ by roots.