1. Use the Weyl dimension formula to list all the (complex continuous) irreducible representations of the compact connected Lie group of type $G_2$, of dimension at most 200.

The root system of $G_2$ can be written in many ways. I have forgotten exactly what I did in class, but one possibility is

$$X^* = \{ \lambda \in \mathbb{Z}^3 \mid \sum \lambda_j = 0 \}, \quad X_* = \{ \xi \in \mathbb{Q}^3 \mid \sum \xi_j = 0, \xi - \xi_j \in \mathbb{Z} \}$$

$$R = \{ e_i - e_j \mid i \neq j \} \cup \{ \pm (2e_p - e_q - e_r) \mid \{ p, q, r \} = \{1, 2, 3 \} \}$$

$$R^\vee = \{ e_i - e_j \mid i \neq j \} \cup \{ \pm \frac{1}{3} (2e_p - e_q - e_r) \mid \{ p, q, r \} = \{1, 2, 3 \} \};$$

the coroot $\alpha^\vee$ appears always directly below $\alpha$ in these lists. We can use the regular element $(3, -1, -2) \in X_*$ to define positive roots; then

$$R^+ = \{ e_1 - e_2, e_2 - e_3, e_1 - e_3 \} \cup \{ 2e_1 - e_2 - e_3, e_1 - 2e_2 + e_3, e_1 + e_2 - 2e_3 \},$$

$$\rho = (1, 0, -1) + (2, -1, -1) = (3, -1, -2).$$

Here $\rho$ is half the sum of the positive roots; I’ve computed it by taking half the sum of the first three listed positive roots and adding it to half the sum of the last three listed. The simple roots (those not equal to a sum of other positive roots) are

$$\Pi = \{ \alpha = (1, -2, 1), \beta = (0, 1, -1) \},$$

$$\Pi^\vee = \{ \alpha^\vee = \frac{1}{3} (1, -2, 1), \beta^\vee = (0, 1, -1) \};$$

by inspection, the positive coroots are

$$(R^\vee)^+ = \{ \alpha^\vee, \beta^\vee, \alpha^\vee + \beta^\vee, 2\alpha^\vee + \beta^\vee, 3\alpha^\vee + \beta^\vee, 3\alpha^\vee + 2\beta^\vee \}. $$

It’s also clear by inspection that the coroots $\alpha^\vee$ and $\beta^\vee$ are a basis of $X_*$. It follows that the dominant elements $\xi$ of $X^*$ may be indexed by the two nonnegative integers

$$n = (\xi, \alpha^\vee), \quad m = (\xi, \beta^\vee).$$

The problem asks: **for which pairs $(n, m)$ of nonnegative integers is the dimension of $\pi(\xi)$ at most 200?** Of course we can use the Weyl dimension formula

$$\dim \pi(\xi) = \prod_{\gamma \in R^+} \langle \xi + \rho, \gamma^\vee \rangle / \langle \rho, \gamma^\vee \rangle.$$ 

We compute everything by writing the coroots $\gamma^\vee$ in terms of $\alpha^\vee$ and $\beta^\vee$ as above. Noting that

$$\langle \rho, \alpha^\vee \rangle = 1 = \langle \rho, \beta^\vee \rangle,$$

we get

$$\dim \pi(\xi) = \frac{(n + 1)(m + 1)(n + m + 2)(2n + m + 3)(3n + m + 4)(3n + 2m + 5)}{1 \cdot 1 \cdot 2 \cdot 3 \cdot 4 \cdot 5}$$

That is, we need nonnegative $n$ and $m$ so that

$$(n + 1)(m + 1)(n + m + 2)(2n + m + 3)(3n + m + 4)(3n + 2m + 5)/120 \leq 200.$$

It’s clear that the formula is strictly increasing as a function of $n$ and as a function of $m$ separately.
We begin with \( n = 0 \), and see what the possibilities are for \( m \); the inequality is
\[
(m + 1)(m + 2)(m + 3)(m + 4)(2m + 5)/120 \leq 200.
\]
The left side is an increasing function of \( m \), and the first few values are
\[
1 \ (m = 0), \quad 7 \ (m = 1), \quad 27 \ (m = 2), \quad 77 \ (m = 3), \quad 182 \ (m = 4);
\]
the next is 378, and from then on they exceed 200. Next, we take \( n = 1 \). Now the condition is
\[
2(m + 1)(m + 3)(m + 5)(m + 7)(2m + 8)/120 \leq 200,
\]
or
\[
(m + 1)(m + 3)(m + 4)(m + 5)(m + 7)/30 \leq 200.
\]
This time the first values are
\[
14 \ (m = 0), \quad 64 \ (m = 1), \quad 189 \ (m = 2), \quad 448 \ (m = 3),
\]
and from then on they exceed 200. Next, we take \( n = 2 \). Now the condition is
\[
3(m + 1)(m + 4)(m + 7)(m + 10)(2m + 11)/120 \leq 200,
\]
or
\[
(m + 1)(m + 4)(m + 7)(m + 10)(2m + 11)/40 \leq 200.
\]
The first two values are
\[
77 \ (m = 0), \quad 286 \ (m = 1),
\]
and from then on they exceed 200. For \( n = 3 \) and \( m = 0 \), the value is 273; so for \( n \geq 3 \), the dimension exceeds 200. So the nine values of \( m \) and \( n \) tabulated above are all those for which the dimension is at most 200. Dimension 77 appears twice, for \((3, 0)\) and \((0, 2)\).

2. Suppose \( G \) is a compact connected Lie group with maximal torus \( T \), and \( \xi \in X^*(T) \). Write \( \pi(\xi) \) for the unique irreducible representation of \( G \) of extremal weight \( \xi \). Show that the character \( \Theta(\xi) \) of \( \pi(\xi) \) is real-valued if and only if \( \xi \) is conjugate to \(-\xi\) under the Weyl group \( W(G, T) \).

Because each character \( \xi \) takes values in the unit circle, its complex conjugate is equal to its inverse; that is (in the exponential notation used in class)
\[
e^{\xi} = e^{-\xi}.
\]
If \( \rho \) is half the sum of a set \( R^+ \) of positive roots, then \(-\rho\) is equal to half the sum of \(-R^+ = w_0R^+\); here \( w_0 \) is the (unique) element of \( W(R) \) carrying \( R^+ \) to \(-R^+\). If we write
\[
\epsilon: W \to \{\pm 1\}, \quad \epsilon(s_\alpha) = -1
\]
then the Weyl denominator (a function on the “\( \rho \) double cover of \( T \)” discussed in class)
\[
D = e^\rho \prod_{\alpha \in R^+}(1 - e^{-\alpha})
\]
satisfies
\[
w \cdot D = \epsilon(w)D \quad (w \in W).
\]
It follows that
\[
\overline{D} = w_0 \cdot D = \epsilon(w_0)D.
\]
Similarly, if ξ is any dominant weight, then ξ + ρ is dominant regular, so
\[ R^+ = \{ \alpha \in R \mid \langle \xi + \rho, \alpha^\vee \rangle > 0 \} \]
In the same way \(-\xi - \rho\) defines \(-R^+ = w_0 R^+\). It follows that ξ is conjugate under W to \(-\xi\) if and only if \(\xi = -w_0 \xi\).

The Weyl numerator
\[ N(\xi) = \sum_{w \in W} e(w) e^{w(\xi + \rho)} \]
satisfies
\[ \overline{N(\xi)} = \epsilon(w_0) N(-w_0 \xi). \]
Using the Weyl character formula, we get
\[ \Theta(\xi) = \Theta(-w_0 \xi). \]
That is, \(\Theta(\xi)\) is real-valued if and only if \(\xi = -w_0 \xi\); that is, if and only if ξ is conjugate to \(-\xi\) by W.

3. List the dominant weights of the 64-dimensional irreducible representation of \(G_2\).

According to the calculation in Problem 1, the unique irreducible representation of dimension 64 has dominant extremal weight \(\xi\) indexed by (1, 1); that is,
\[ \langle \xi, \alpha^\vee \rangle = 1, \quad \langle \xi, \beta^\vee \rangle = 1. \]
We can calculate this \(\xi\) by solving a system of linear equations; the answer is
\[ \xi = (3, -1, -2) = 3\alpha + 5\beta. \]
According to Proposition 1.4 in the notes 757repweights.pdf on the course website, the dominant weights appearing in \(\pi(\xi)\) are precisely the dominant weights of the form
\[ \mu = m\alpha + n\beta, \quad 0 \leq m \leq 3, \quad 0 \leq n \leq 5 \quad (m, n \in \mathbb{Z}). \]
The condition “dominant” has two parts:
\[ \langle \mu, \alpha^\vee \rangle \geq 0, \quad \text{or} \quad 2m - n \geq 0, \]
(since \(\langle \alpha, \alpha^\vee \rangle = 2\) and \(\langle \beta, \alpha^\vee \rangle = -1\)) and
\[ \langle \mu, \beta^\vee \rangle \geq 0, \quad \text{or} \quad -3m + 2n \geq 0. \]
These two inequalities may be written as
\[ 4m \geq 2n \geq 3m. \]
So we list the solutions. If \(m = 0\), then \(n = 0\). If \(m = 1\), then \(n = 2\). If \(m = 2\), then \(n = 3\) or 4. If \(m = 3\), then \(n = 5\) or \(n = 6\); and the latter is ruled out by the requirement \(n \leq 5\). Therefore there are exactly five dominant weights of \(\pi(\xi)\):
\[ 0\alpha + 0\beta = (0, 0, 0), \quad \alpha + 2\beta = (1, 0, -1), \quad 2\alpha + 3\beta = (2, -1, -1), \]
\[ 2\alpha + 4\beta = (2, 0, -2), \quad 3\alpha + 5\beta = (3, -1, -2). \]