These problems use the notes on classical groups on the class website. In particular, the notes include a proof of what had been labeled "problem 1" when I wrote these on the board; so that one has been removed. I have added a new problem 4, which amounts to a hint for problem 5.

1. Suppose \((\pi, V_{\pi})\) is a finite-dimensional irreducible of \(O(2p+1)\). Prove that the restriction of \(\pi\) to \(SO(2p+1)\) is still irreducible. (This is true for representations over any field, and no continuity hypothesis is needed. But you can prove it only for continuous complex representations if you like.)

If \(G\) is any group and \(Z\) is the center of \(G\), then the linear transformations \(\pi(z)\) (for \(z \in Z\)) commute with \(\pi(G)\). If \(\pi\) is irreducible, this means that they belong to the division algebra \(D_{\pi}\). The resulting group homomorphism \(\chi_{\pi}: Z \to D_{\pi}\) is called the central character of \(\pi\).

The center of \(O(n)\) the two-element group \(Z = \{\pm I\}\). The central character must carry \(-I\) to an element of \(D_{\pi}\) whose square is 1. The only such elements (even in characteristic two!) are \(\pm 1\). (In characteristic two there is only one.) Therefore \(\pi(Z)\) is either trivial or \(\{\pm I_{V_{\pi}}\}\).

If \(n = 2p + 1\), \(-I_{2p+1}\) has determinant \(-1\). It follows immediately that \(O(2p+1) \cong SO(2p+1) \times Z\). Since \(Z\) acts by \(\pm I\), any subspace invariant under \(SO(2p+1)\) remains invariant under all of \(O(2p+1)\), as we wished to show.

2. The notes in (4.3) define a homomorphism from \(U(n)\) to \(SO(2n)\) using the standard identification \(C \cong \mathbb{R}^2\). If \(\epsilon = (\epsilon_1, \ldots, \epsilon_n) \in \{0, 1\}^n\),

then we can get another identification

\[
\begin{align*}
\mathbb{C}^n &\xrightarrow{\phi_{\epsilon}} \mathbb{R}^{2n}, \\
z_p &\mapsto \begin{cases} (a_p, b_p) & \epsilon_p = 0 \\
(a_p, -b_p) & \epsilon_p = 1. \end{cases}
\end{align*}
\]

That is, we replace \(z_p\) by \(z_p\) whenever \(\epsilon_p = 1\). The identification \(\phi_{\epsilon}\) defines a homomorphism

\[
j_{\epsilon}: U(n) \to O(2n).
\]

Define

\[
J_{\epsilon} = j_{\epsilon}(i I_n) \in O(2n).
\]

Suppose 0 \(\leq p \leq n\) is an integer, and \(n = p + q\). Write

\[
\epsilon(p, q) = (0, \ldots, 0, 1, \ldots, 1) \quad (p \text{ zeros and } q \text{ ones}).
\]

a) Show that the image of \(j_{\epsilon}\) is equal to the centralizer in \(O(2n)\) of the element \(J_{\epsilon}\).

A complex linear transformation is the same thing as a real-linear transformation that commutes with multiplication by \(i\). It follows that (if we
extend the definitions appropriately) the image \( j_\epsilon(GL(n, \mathbb{C})) \) is equal to the centralizer in \( GL(2n, \mathbb{R}) \) of \( J_\epsilon \).

An element of \( U(n) \) is an element of \( GL(n, \mathbb{C}) \) that preserves length of vectors in \( \mathbb{C}^n \). An element of \( O(2n) \) is an element of \( GL(2n, \mathbb{R}) \) that preserves length of vectors in \( \mathbb{R}^{2n} \). The identification \( \phi_\epsilon \) of \( \mathbb{C}^n \) with \( \mathbb{R}^{2n} \) preserves length of vectors.

The claim follows from these two paragraphs.

b) **Show that the centralizer of** \( J_{(n,0)}J_{(p,q)} \) **in** \( O(2n) \) **is equal to** \( O(2p) \times O(2q) \).

The matrix of \( J_\epsilon \) consists of \( 2 \times 2 \) blocks on the diagonal

\[
\begin{pmatrix}
0 & -1 \\
1 & 0
\end{pmatrix} \quad (\epsilon_p = 0),
\begin{pmatrix}
0 & 1 \\
-1 & 0
\end{pmatrix} \quad (\epsilon_p = 1).
\]

It follows that \( J_{(n,0)}J_{(p,q)} \) consists of \( p \times 2 \times 2 \) diagonal blocks \(-I_2\), and \( q \times 2 \times 2 \) diagonal blocks \( I_2\); that is

\[
J_{(n,0)}J_{(p,q)} = \begin{pmatrix}
-I_{2p} & 0 \\
0 & I_{2q}
\end{pmatrix}.
\]

The centralizer of this matrix consists of all matrices of the form

\[
\begin{pmatrix}
A & 0 \\
0 & B
\end{pmatrix},
\]

with \( A \) and \( B \) square of sizes \( 2p \) and \( 2q \) respectively. The claim follows.

c) **Show that**

\[
\begin{array}{c}
\text{cent in } O(2p) \text{ of } J_{(p,0)} \\
\text{and } J_{(p,0)}
\end{array} \times \begin{array}{c}
\text{cent in } O(2q) \text{ of } J_{(q,0)} \\
\text{and } J_{(q,0)}
\end{array}.
\]

(The original problem mistakenly wrote \( J \) in the last formula.) We need to compute the common centralizer of the two matrices \( J_{(n,0)} \) and \( J_{(p,q)} \). This centralizer is contained in the centralizer of the product matrix, which was computed in (b) to be \( O(2p) \times O(2q) \). Since the matrices \( J \) respect this product decomposition, it follows that the intersection is

\[
\begin{array}{c}
\text{cent in } O(2p) \text{ of } J_{(p,0)} \\
\text{and } J_{(p,0)}
\end{array} \times \begin{array}{c}
\text{cent in } O(2q) \text{ of } J_{(q,0)} \\
\text{and } J_{(q,0)}
\end{array}.
\]

This is

\[
\begin{array}{c}
\text{cent in } O(2p) \text{ of } J_{(p,0)} \\
\text{and } J_{(p,0)}
\end{array} \times \begin{array}{c}
\text{cent in } O(2q) \text{ of } \pm J_{(q,0)} \text{ if } J_{(q,0)} \text{ is in } O(2q), \\
\text{or } O(2q) \text{ if } J_{(q,0)} \text{ is in } O(2q)
\end{array},
\]

which is what we want by (a).

d) **Now take** \( n = 2 \). **Show that**

\[
j_{(2,0)}(SU(2)) \cap j_{(1,1)}(SU(2)) = \{ \pm I_4 \} \subset SO(4).
\]

According to (c),

\[
j_{(2,0)}(U(2)) \cap j_{(1,1)}(U(2)) = [j_{(1,0)}U(1)] \times [j_{(0,1)}U(1)]
\]

\[
= \begin{pmatrix}
r(\phi_1) & 0 \\
0 & r(-\phi_2)
\end{pmatrix}.
\]
Here \( r(\phi) \) is the \( 2 \times 2 \) orthogonal matrix of the rotation through an angle \( \phi \). The indicated matrix in \( SO(4) \) is equal to

\[
j_{\epsilon(2,0)}(e^{i\phi_1} \begin{pmatrix} 0 & 0 \\ 0 & e^{i\phi_2} \end{pmatrix}) = j_{\epsilon(1,1)}(e^{i\phi_1} \begin{pmatrix} 0 & 0 \\ 0 & e^{-i\phi_2} \end{pmatrix}).
\]

The change in the sign (complex conjugation) on the \( \phi_2 \) term arises because in the first case we are using the isomorphism \( j_0 : U(1) \to SO(2) \) and in the second case \( j_1 \). The problem asks us to restrict to complex matrices of determinant 1; that is, to \( \phi_1 + \phi_2 \in 2\pi\mathbb{Z} \) in the first case, and to \( \phi_1 - \phi_2 \in 2\pi\mathbb{Z} \) in second. The simultaneous solutions of these two requirements are

\[
\phi_j \in \pi\mathbb{Z}, \quad \phi_1 + \phi_2 \in 2\pi\mathbb{Z}.
\]

Equivalently,

\[
e^{i\phi_j} = \pm 1, \quad e^{i\phi_1}e^{i\phi_2} = 1.
\]

Because rotation by \( \pi \) is \(-I_2\), the matrices in \( SO(4) \) are

\[
\pm I_2 \begin{pmatrix} 0 & 0 \\ 0 & \pm I_2 \end{pmatrix},
\]

and the two signs must be the same. This is what we wished to show.

(e) Show that \( j_{\epsilon(2,0)}(SU(2)) \) and \( j_{\epsilon(1,1)}(SU(2)) \) commute with each other. (I don’t know an easy way to see this. One possibility is to look at the Lie algebras.)

(Proof to be added later.)

f) Conclude that

\[
SO(4) \simeq [SU(2) \times SU(2)]/\{\pm I_2\},
\]

the two-element subgroup embedded diagonally in the product.

Because of (e), the maps in (d) define an injective Lie group homomorphism from the right side to the left. We know (for example from the classical groups notes) that \( SU(2) \) has real dimension \( 2^2 - 1 = 3 \), and \( SO(4) \) has real dimension \( \binom{4}{2} = 6 \). Therefore the image of the right side has the same dimension as the left side. By Lie theory, the image is open. Since \( SO(4) \) is connected, any open subgroup is equal to \( SO(4) \); so the map is bijective. Since \( SU(2) \) is compact, it follows that the map is a homeomorphism.

3. Write \( H_m(n) \) for the complex representation of \( O(n) \) on complex-valued harmonic polynomials of degree \( m \). Recall that \( H_m(n) \) is irreducible or zero, of dimension \( \binom{m+n-1}{n-1} - \binom{m+n-3}{n-1} \), and that

\[
H_m(n)|_{O(n-1)} = \sum_{p=0}^{m} H_p(n-1)
\]

for \( n \geq 2 \). In particular,

\[
\dim H_m(O(4)) = (m+1)^2,
\]

\[
H_m(4)|_{O(3)} = H_m(3) \oplus H_{m-1}(3) \oplus \cdots \oplus H_0(3).
\]

The formula for dimensions corresponding to the last statement is

\[
(m+1)^2 = (2m+1) + (2m-1) + \cdots + 1.
\]
Show that $H_m(4)$ remains irreducible on restriction to $SO(4)$.

The subgroup $SO(4)$ has index two in $O(4)$ By Clifford theory discussed in class, the only alternative to irreducibility is that the restriction to $SO(4)$ is a direct sum of two irreducible representations interchanged by the action of $O(4)/SO(4)$ on irreducibles of $SO(4)$. Because the component group is generated by the matrix

$$
\begin{pmatrix}
I_3 & 0 \\
0 & -1
\end{pmatrix}
$$

which centralizes $SO(3)$, it follows that in this case the two irreducibles of $SO(4)$ would have to have isomorphic restrictions to $SO(3)$.

On the other hand, problem 1 says that the restriction of $H_m(4)$ to $SO(3)$ is the sum of irreducible representations of dimensions 1, 3, ..., $2m + 1$. Since the dimensions are distinct, these representations are all inequivalent; so they cannot be partitioned into two isomorphic sets. Therefore the reducible restriction case is impossible.

4. The inclusion $j_{(2,0)}$ of $SU(2)$ in $SO(4)$, and equations (4.4) in the notes, define an action of the unit quaternions on the sphere in $\mathbb{R}^4$; that is, on the unit quaternions. Show that this action is left multiplication. Show that the corresponding action defined using $j_{(1,1)}$ is right multiplication by the inverse. (Recall that the inverse of a unit quaternion is equal to its conjugate.)

According to the notes, the unit quaternion $z + wj$ acts on $\mathbb{C}^2$ (through the isomorphism with $SU(2)$) by the matrix

$$
\begin{pmatrix}
z & -w \\
w & z
\end{pmatrix}.
$$

The isomorphism of $\mathbb{C}^2$ with $\mathbb{H}$ is

$$
\begin{pmatrix}
a \\
b
\end{pmatrix} \mapsto a + jb.
$$

(The reason the $b$ appears on the left is that we regard $\mathbb{H}$ as a right vector space over $\mathbb{C}$, and therefore by restriction as a right vector space over $\mathbb{C}$. The action of $SU(2)$ sends the basis vector $(1 \ 0)$ to $(z \ w)$; that is, it sends the quaternion 1 to $z + wj$, which is left multiplication by $z + wj$. Similarly, it sends $j$ to $-w + jz = -w + zj = (z + wj)j$.

This proves that the first $Sp(1) \simeq SU(2)$ is indeed acting by left multiplication on $\mathbb{H}$. The calculation for the second $SU(2)$ is similar.

5. Show that $SU(2)$ has exactly one irreducible continuous complex representation $V_m$ of dimension $m$ for every integer $m \geq 1$.

According to the Peter-Weyl theorem,

$$
L^2(SU(2)) = \bigoplus_{\text{irreducible}} V_\pi \otimes V_\pi^*;
$$

this is the decomposition into irreducible representations of $SU(2) \times SU(2)$. According to problems 3 and 4,

$$
L^2(SU(2)) = \bigoplus_{m \geq 1} H_{m-1}(4),
$$

with $H_{m-1}(4)$ an irreducible representation of $SU(2) \times SU(2)$ of dimension $m^2$. The claim follows by comparing these two formulas.