1. Write $S^{n-1}$ for the unit sphere in $\mathbb{R}^n$, and $O(n)$ for the group of $n \times n$ real orthogonal matrices. Write

$$S^k(\mathbb{C})^n = \text{complex polynomial fns on } \mathbb{R}^n, \text{ homog of degree } k.$$ 

Write

$$V = C(S^{n-1})$$

for the continuous complex-valued functions on the sphere, and $V_{\text{even}}$ (respectively $V_{\text{odd}}$ for the subspace of even (respectively odd) functions.

a) Show that restriction to the sphere defines inclusions

$$S^{2m}(\mathbb{C})^n \hookrightarrow V_{\text{even}}, \quad S^{2m+1}(\mathbb{C})^n \hookrightarrow V_{\text{odd}},$$

for all $m \geq 0$. Write $W^k$ for the image of $S^k(\mathbb{C})^n$; we discussed in class the fact that $W^k$ is an $O(n)$-invariant subspace of $V$.

b) Show that $W^k \subset W^{k+2}$.

c) Show that the sum over $k$ of $W^k$ is a dense subspace of $V$; that is, that for any continuous function $f$ on the sphere, and any $\epsilon > 0$, there is an $m \geq 0$ and a function $h \in W^{2m} + W^{2m+1}$ such that $|h - f| < \epsilon$.

d) Write

$$\Delta = \frac{\partial^2}{\partial x_1^2} + \cdots + \frac{\partial^2}{\partial x_n^2}$$

for the Laplace operator. Obviously $\Delta$ defines a linear map

$$\Delta: S^k(\mathbb{C})^n \rightarrow S^{k-2}(\mathbb{C})^n,$$

(almost obviously) respecting the action of $O(n)$. Define

$$H^k = \ker(\Delta|_{S^k(\mathbb{C})^n}),$$

the space of harmonic polynomials of degree $k$. Similarly, we have a linear map

$$r^2: S^{k-2}(\mathbb{C})^n \rightarrow S^k(\mathbb{C})^n,$$

also respecting the action of $O(n)$. Prove that

$$S^k \simeq \text{im}(r^2) \oplus H^k.$$ 

(Hint: one approach is to recall that if $T: E \rightarrow F$ is a linear map between finite-dimensional inner product spaces, then

$$F = \text{im}(T) \oplus \ker(T^*)$$

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It follows from the problem that
\[ W^k \simeq H^k \oplus H^{k-2} \oplus H^{k-4} \ldots, \]
an \( O(n) \)-invariant direct sum decomposition with \( [k/2] \) terms, and that
\[ \dim H^k = \binom{n + k - 1}{k} - \binom{n + k - 3}{k - 2}. \]

2. (With thanks to Inna Entova-Aizenbud.)
   a) Prove (as claimed in class) that for \( n = 2 \), \( H^k \) (restricted to the unit circle) is spanned by \( \cos(k\theta) \) and \( \sin(k\theta) \).
   b) I said in class that \( H^k \) consists of even functions if \( k \) is even and odd functions if \( k \) is odd. But \( \cos(k\theta) \) is always even, and \( \sin(k\theta) \) is always odd. Explain.