18.755 ninth problems solutions

I described in class how to find all connected groups with a given Lie algebra from a simply connected group and knowledge of its center. So this problem set is about finding centers. Calculate the center means identify the group elements in the center, and say what the group law is. In some cases you may not be able to give a complete answer; say as much as you can.

Recall that $SO(n)$ is the group of $n \times n$ real orthogonal matrices of determinant 1. We proved in class that $SO(n)$ is connected for all $n \geq 1$, and that

$$
\pi_1(SO(n)) = \begin{cases} 
0 & (n = 1) \\
\mathbb{Z} & (n = 2) \\
\mathbb{Z}/2\mathbb{Z} & (n \geq 3)
\end{cases}.
$$

This fundamental group has a unique quotient of order 2 for $n \geq 2$, and therefore there is a unique connected double cover $Spin(n)$ ($n \geq 2$) with a short exact sequence of Lie groups

$$1 \to \{1, \epsilon\} \to Spin(n) \to SO(n) \to 1.$$ 

1. Calculate the center of $SO(n)$ for all $n \geq 1$.

For $n = 1$, $SO(1)$ is the trivial group with trivial center.

For $n = 2$, $SO(2)$ is the group of rotations of the plane, isomorphic to the abelian group $S^1$ of unit complex numbers, by the map

$$
\begin{pmatrix} 
\cos \theta & \sin \theta \\
-\sin \theta & \cos \theta
\end{pmatrix} \mapsto e^{i\theta}.
$$

$$Z(SO(2)) = SO(2).$$

For $n \geq 3$, it’s helpful to use

**Proposition.** Suppose $n \geq 2$, and that $p$ and $q$ are distinct integers between 1 and $n$. Define $D(p, q)$ to be the diagonal matrix with entries $-1$ in rows $p$ and $q$, and $+1$ in all other rows. Then

$$AD(p, q) = D(p, q)A \iff A_{pj} = A_{qj} = A_{jp} = A_{jq} = 0, \quad (j \notin \{p, q\}).$$

The reason is that multiplication on the left by $D(p, q)$ multiplies rows $p$ and $q$ (the entries $A_{px}$ and $A_{qx}$ by $-1$; and multiplication on the right by $D(p, q)$ multiplies columns $p$ and $q$ (entries $A_{xp}$ and $A_{xq}$) by $-1$.

Similarly,
Proposition. Suppose $n \geq 2$, and that $p$ and $q$ are distinct integers between 1 and $n$. Define $S(p,q)$ to be the signed transposition matrix

$$S(p,q)_{pp} = S(p,q)_{qq} = 0, \quad S(p,q)_{pq} = 1, \quad S(p,q)_{qp} = -1, \quad S(pq)_{jj} = 1 \ (j \notin \{p,q\}),$$

and all other entries are zero. For any diagonal matrix $D$,

$$DS(p,q) = S(p,q)D \iff D_{pp} = D_{qq}.$$  

Since $SO(n)$ includes all the diagonal matrices $D_{pq}$, it follows that $Z(SO(n))$ must consist of matrices $Z$ commuting with all these $D(p,q)$. Consequently the first proposition implies that any off-diagonal entry of $Z_{pq}$ must be zero. (The reason is that since $n \geq 3$, we can find a third integer $q'$ between 1 and $n$ distinct from $p$ and $r$.) So $Z$ is diagonal. In addition, $SO(n)$ includes all the matrices $S(p,q)$. (We included the sign to make $\det S(p,q)$ equal to +1.) So $Z$ must commute with all the $S(p,q)$, so the second proposition implies that $Z$ must be a scalar $zI$.

Since we are looking at real matrices, $z$ is real. Since $\det Z = 1, z^n = 1$. So $z = 1$ (if $n$ is odd) or $\pm 1$ (if $n$ is even). Conclusion is

$$Z(SO(n)) = \{I_n\} \quad (n \text{ odd}),$$

$$Z(SO(n)) = \{\pm I_n\} \quad (n \geq 4 \text{ even}).$$

2. Calculate the center of $\text{Spin}(n)$ for all $n \geq 2$.

We use the exact sequence

$$1 \to \{1, \epsilon\} \to \text{Spin}(n) \to \text{SO}(n) \to 1$$

explained before the problems. Write $\pi: \text{Spin}(n) \to \text{SO}(n)$ for the projection. Since $\text{Spin}(n)$ is a connected Lie group,

$$Z(\text{Spin}(n)) = \ker(\text{Ad});$$

Here $\text{Ad}: \text{Spin}(n) \to \text{Aut}(g)$ is the map discussed in class. This adjoint action factors through $\pi$:

$$\text{Ad}_{\text{Spin}(n)} = \text{Ad}_{\text{SO}(n)} \circ \pi.$$  

The last two equations imply that

$$Z(\text{Spin}(n)) = \pi^{-1}(Z(\text{SO}(n))).$$

We therefore get a short exact sequence of groups

$$1 \to \{1, \epsilon\} \to Z(\text{Spin}(n)) \to Z(\text{SO}(n)) \to 1.$$
Now we apply the result of Problem 1. This tells us

\[ Z(\text{Spin}(2)) = \text{Spin}(2), \]

a double cover of the circle and therefore again a circle group. Also

\[ Z(\text{Spin}(n)) = \{1, \epsilon\} \quad (n \geq 3 \text{ odd}). \]

The case of \( n \geq 4 \) even is harder. We get

\[ 1 \to \{1, \epsilon\} \to Z(\text{Spin}(2m)) \to \{\pm I_{2m}\} \to 1, \quad \text{ALMOSTDONE} \]

which tells us that

\[ |Z(\text{Spin}(2m))| = 4 \quad (m \geq 2). \]

But \textit{which} group of order 4 is it? Answer is

\[ Z(\text{Spin}(2m)) \simeq \begin{cases} \mathbb{Z}/2\mathbb{Z}^2 & (m \geq 4 \text{ even}) \\ \mathbb{Z}/4\mathbb{Z} & (m \geq 4 \text{ odd}) \end{cases}. \]

A bit more precisely, if \( m \) is even, then either preimage \( \tilde{I}_{2m} \) of \( I_{2m} \) has order two, and together with \( \epsilon \) generates the center. If \( m \) is odd, then either preimage \( \tilde{I}_{2m} \) of \( I_{2m} \) satisfies

\[ [\tilde{I}_{2m}]^2 = \epsilon, \]

and so \( \tilde{I}_{2m} \) generates \( Z(\text{Spin}(2m)) \).

Here is why. Recall from Problem Set 5 that one way to realize the universal cover \( \tilde{G} \) of a connected Lie group \( G \) is as homotopy classes of paths \( \gamma \) in \( G \) starting at \( e \). The projection map to \( G \) is \( \pi(\gamma) = \gamma(1) \). We can therefore define a preimage \( \gamma(2p - 1, 2p) \) of the diagonal element \( D(2p - 1, 2p) \) of \( G \) by

\[
\gamma(2p - 1, 2p)(t) = \begin{pmatrix}
I_{2(p-1)} & \cos \pi t & \sin \pi t \\
0 & 0 & 0 \\
-\sin \pi t & \cos \pi t & I_{2(m-p)}
\end{pmatrix} \quad (0 \leq t \leq 1)
\]

for \( 1 \leq p \leq m \). Obviously all of these \( \gamma(2p - 1, 2p)(t) \) commute with each other, and their product

\[ \gamma(1, 2) \cdots \gamma(2m - 1, 2m) = \text{def} \quad \tilde{I}_{2m} \]

is a preimage in \( \tilde{G} \) of \( -I_{2m} \). Now it is clear from the exact sequence (ALMOSTDONE) that \( Z(\text{Spin}(2m)) \) is generated by \( \epsilon \) and \( -I_{2m} \). We just need to understand what group they generate.
Clearly $\gamma(2p - 1, 2p)^2$ is the standard loop going once around the block diagonal $SO(2)$ on coordinates $2p - 1$ and $2p$. It is therefore a generator of $\pi_1(SO(2))$. Because of the way we proved in class that $\pi_1(SO(n))$ is $\mathbb{Z}/2\mathbb{Z}$ for $n \geq 3$, this $\pi_1(SO(2))$ generator must map to $\epsilon \in \pi_1(SO(2m))$. That is

$$\gamma(2p - 1, 2p)^2 = \epsilon \in \text{Spin}(2m).$$

Multiplying these $m$ equations together (remembering that all the $\gamma(2p - 1, 2p)$ commute with each other) gives

$$-I_{2m}^2 = \epsilon^m = \begin{cases} 1 & m \text{ even} \\ \epsilon & m \text{ odd}. \end{cases}$$

This is what we claimed above.

3. Calculate the center of the real Lie group $SU(n)$ (consisting of $n \times n$ complex unitary matrices of determinant 1).

One can use minor variants of the two propositions in the solution of Problem 2; but instead I will just lazily copy an old (and very similar) solution.

It is easiest first to calculate

$$Z = \text{all } n \times n \text{ matrices commuting with } SU(n);$$

then the center $Z(G)$ is just $Z \cap G$. What makes this easier is that $Z$ is an algebra, and in particular a vector space over $\mathbb{C}$. It may also be described as

$$Z = \text{matrices commuting with } \mathbb{C}\text{-linear combinations of elements of } SU(n).$$

Because $SU(1)$ is the trivial group, equal to its own center, we may assume $n \geq 2$. Now $SU(n)$ includes the four linearly independent matrices

$$\begin{pmatrix} -1 & 0 & 0_{1 \times (n-2)} \\ 0 & -1 & 0_{1 \times (n-2)} \\ 0_{(n-2) \times 1} & 0_{(n-2) \times 1} & I_{n-2} \end{pmatrix}, \quad \begin{pmatrix} i & 0 & 0_{1 \times (n-2)} \\ 0 & -i & 0_{1 \times (n-2)} \\ 0 & 0 & I_{n-2} \end{pmatrix},$$

$$\begin{pmatrix} 0 & 1 & 0_{1 \times (n-2)} \\ -1 & 0 & 0_{1 \times (n-2)} \\ 0_{(n-2) \times 1} & 0_{(n-2) \times 1} & I_{n-2} \end{pmatrix}, \quad \begin{pmatrix} 0 & i & 0_{1 \times (n-2)} \\ i & 0 & 0_{1 \times (n-2)} \\ 0_{(n-2) \times 1} & 0_{(n-2) \times 1} & I_{n-2} \end{pmatrix}.$$

Taking complex linear combinations, we can therefore find the four matrices

$$e_{pq} \quad 1 \leq p, q \leq 2.$$

The matrices commuting with these four are just

$$\begin{pmatrix} aI_2 & 0 \\ 0 & B \end{pmatrix} \quad (a \in \mathbb{C}, B \text{ an } (n-2) \times (n-2) \text{ matrix}).$$
Using other pairs of coordinates in the same way, we conclude that

\[ Z \subset \{ aI_n \mid a \in \mathbb{C} \}, \]

the algebra of \( n \times n \) scalar matrices. Of course scalar matrices do commute with everything, so \( Z \) consists of scalar matrices.

Therefore

\[ Z(SU(n)) = Z \cap SU(n) = \{ aI_n \mid a \in \mathbb{C}, a\overline{a} = 1, a^n = 1 \}. \]

(The first condition is unitarity, and the second is determinant one.) Therefore

\[ \text{center of } SU(n) = \{ \text{n-th roots of unity} \} = \{ e^{2\pi ij/n} \mid j \in \mathbb{Z}/n\mathbb{Z} \} \simeq \mathbb{Z}/n\mathbb{Z}, \]

a cyclic group of order \( n \). Even though the proof used \( n \geq 2 \), the answer agrees with the answer we got for \( n = 1 \).

If you work in the analogous world of algebraic groups, it is important to remember that for \( n \geq 3 \), the isomorphism of \( n \)-th roots of 1 with \( \mathbb{Z}/n\mathbb{Z} \) is not canonical: it is not preserved by automorphisms of the field \( \mathbb{C} \), for example. It is not even preserved by the continuous automorphism \( z \mapsto \overline{z} \) (which induces the inversion automorphism of \( \mathbb{Z}/n\mathbb{Z} \); this issue can affect questions about Lie groups); but for algebraic groups you need to worry also about discontinuous automorphisms, and these can induce any automorphism of \( \mathbb{Z}/n\mathbb{Z} \).

4. Calculate the center of \( Sp(n) \) (consisting of \( n \times n \) quaternionic matrices preserving the standard Hermitian form on \( \mathbb{H}^n \)).

The answer is

\[ \text{center of } Sp(n) = \{ \pm I_n \} \simeq \mathbb{Z}/2\mathbb{Z}. \]

There is a proof exactly parallel to that given for Problem 3, using the real algebra

\[ Z = \text{quaternionic matrices commuting with } \mathbb{R}-\text{linear combinations of elements of } Sp(n). \]

The key facts are

\begin{enumerate}
  \item any \( 2 \times 2 \) quaternionic matrix commuting with \( \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \) must be diagonal; and
  \item any \( 2 \times 2 \) diagonal quaternionic matrix commuting with \( \begin{pmatrix} i & 0 \\ 0 & i \end{pmatrix} \) and \( \begin{pmatrix} j & 0 \\ 0 & j \end{pmatrix} \) must be real; and
  \item any real diagonal \( 2 \times 2 \) matrix commuting with \( \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \) must be scalar.
\end{enumerate}

From these facts we deduce

\[ Z = \text{real scalar matrices}; \]

and the claim about the center follows immediately.
5. You now have three (problem set said four; I intended to include $SO(n)$) infinite families of compact connected Lie groups $\text{Spin}(n_1)$, $SU(n_2)$, and $Sp(n_3)$. Find some (or preferably all!) examples of pairs $(G_1, G_2)$ of groups on these lists satisfying

$$\dim G_1 = \dim G_2, \quad Z(G_1) \cong Z(G_2).$$

You can use the formulas from class

$$\dim \text{Spin}(n_1) = n_1(n_1 - 1)/2, \quad \dim SU(n_2) = n_2^2 - 1, \quad \dim Sp(n_3) = 2n_3^2 + n_3.$$

Write $H_1(n_1) = \text{Spin}(n_1)$, $H_2(n_2) = SU(n_2)$, $H_3(n_3) = Sp(n_3)$. Each of the three sequences of dimensions is strictly increasing, so there can be no pairs of matching dimensions taken from the same sequence. Notice that

$$2n_3^2 + n_3 = (2n_3 + 1)(2n_3)/2,$$

so

$$\dim H_3(n_3) = \dim H_1(2n_3 + 1).$$

Since the center of an odd spin group is $\pm 1$, which is the same as the center of any symplectic group, we get immediately some pairs

$$G_1 = \text{Spin}(2n_3 + 1), \quad G_2 = Sp(n_3).$$

In looking for other coincidences, we will compare only the first two sequences $H_1$ and $H_2$; if we find one involving an odd spin group, then it will immediately give us an additional one involving $Sp$.

So we begin with the number-theoretic problem of solving the equation

$$n_1(n_1 - 1)/2 = (n_2^2 - 1), \quad (NUM)$$

or equivalently

$$n_1(n_1 - 1) = 2(n_2 + 1)(n_2 - 1).$$

Multiplying by 4, this equation become

$$(2n_1 - 1)^2 = 2(2n_2)^2 - 7.$$

The (positive integer) solutions to this equation come in two infinite families, beginning with

$$\begin{pmatrix} n_1 \\ n_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad \begin{pmatrix} n'_1 \\ n'_2 \end{pmatrix} = \begin{pmatrix} 3 \\ 2 \end{pmatrix}.$$

What happens is

if $\begin{pmatrix} x \\ y \end{pmatrix}$ is a solution, so is $\begin{pmatrix} 3x + 4y - 1 \\ 2x + 3y - 1 \end{pmatrix}$. 


The first family of solutions (written as groups)
\[
\left( \begin{array}{c} \text{Spin}(1) \\ \text{SU}(1) \end{array} \right), \quad \left( \begin{array}{c} \text{Spin}(6) \\ \text{SU}(4) \end{array} \right), \quad \left( \begin{array}{c} \text{Spin}(33) \\ \text{SU}(23) \end{array} \right) \ldots
\]

The second is
\[
\left( \begin{array}{c} \text{Spin}(3) \\ \text{SU}(2) \end{array} \right), \quad \left( \begin{array}{c} \text{Spin}(16) \\ \text{SU}(11) \end{array} \right), \quad \left( \begin{array}{c} \text{Spin}(91) \\ \text{SU}(54) \end{array} \right) \ldots
\]

That the indicated process really gives new solutions is high school algebra. That the two series are all the (positive integer) solutions is not too hard, but I will skip it. What we really care about is that the only solutions with \( n_2 < 5 \) are
\[
n_1 = 1, n_2 = 1; \quad n_1 = 3, n_2 = 2; \quad n_1 = 6, n_2 = 4. \quad (\text{SMALLNUM})
\]
(This can be checked by looking at each of the four values of \( n_2 \) separately.) As soon as \( n_2 \geq 5 \), the center of \( H(n_2) \) is a cyclic group of order at least 5. This cannot be the center of any spin group, so the larger solutions to (NUM) do not contribute to solutions to this problem.

The first pair of groups appearing in (SMALLNUM) is
\[
G_1 = \text{Spin}(1), \quad G_2 = \text{SU}(1) = \{1\};
\]
but we have not defined \( \text{Spin}(1) \), so this does not qualify. (There is a standard definition, which gives \( \text{Spin}(1) = \{1, \varepsilon\} \); with this definition, the centers are different, so this is still not a solution.)

The second pair is
\[
G_1 = \text{Spin}(3), \quad G_2 = \text{SU}(2).
\]
The group \( G_1 \) is also paired with \( \text{Sp}(1) \), which we saw was isomorphic to \( \text{SU}(2) \) and to \( \text{Spin}(3) \) in connection with discussing quaternions in class:
\[
\text{Sp}(1) = \text{SU}(2) = \text{Spin}(3).
\]

The third pair is
\[
G_1 = \text{Spin}(6), \quad G_2 = \text{SU}(4).
\]
For each of these groups the center is \( \mathbb{Z}/4\mathbb{Z} \), so they make a solution to the problem. We’ll see later that the groups are isomorphic.

We’ll also see later that \( \text{Spin}(2n+1) \) is isomorphic to \( \text{Sp}(n) \) if and only if \( n = 1 \) or \( n = 2 \).