I described in class how to find all connected groups with a given Lie algebra from a simply connected group and knowledge of its center. So this problem set is about finding centers. Calculate the center means identify the group elements in the center, and say what the group law is. In some cases you may not be able to give a complete answer; say as much as you can.

Recall that \( SO(n) \) is the group of \( n \times n \) real orthogonal matrices of determinant 1. We proved in class that \( SO(n) \) is connected for all \( n \geq 1 \), and that

\[
\pi_1(SO(n)) = \begin{cases} 
0 & (n = 1) \\
\mathbb{Z} & (n = 2) \\
\mathbb{Z}/2\mathbb{Z} & (n \geq 3).
\end{cases}
\]

This fundamental group has a unique quotient of order 2 for \( n \geq 2 \), and therefore there is a unique connected double cover

\[
Spin(n) \quad (n \geq 2)
\]

with a short exact sequence of Lie groups

\[
1 \to \{1, \epsilon\} \to Spin(n) \to SO(n) \to 1.
\]

1. Calculate the center of \( SO(n) \) for all \( n \geq 1 \).

2. Calculate the center of \( Spin(n) \) for all \( n \geq 2 \).

3. Calculate the center of the real Lie group \( SU(n) \) (consisting of \( n \times n \) complex unitary matrices of determinant 1) for all \( n \geq 1 \).

4. Calculate the center of \( Sp(n) \) (consisting of \( n \times n \) quaternionic matrices preserving the standard Hermitian form on \( \mathbb{H}^n \)) for all \( n \geq 1 \).

5. You now have four infinite families of compact connected Lie groups \( Spin(n_1), SU(n_2), \) and \( Sp(n_3) \). Find some (or preferably all!) examples of pairs \((G_1, G_2)\) of groups on these lists satisfying

\[
\dim G_1 = \dim G_2, \quad Z(G_1) \cong Z(G_2).
\]

You can use the formulas from class

\[
\dim SO(n_1) = n_1(n_1 - 1)/2, \quad \dim SU(n_2) = n_2^2 - 1, \quad \dim Sp(n_3) = 2n_3^2 + n_3.
\]