18.755 eighth problem set solutions

Suppose $V$ is an $n$-dimensional vector space over a field $k$ (not of characteristic 2, to keep things simpler) and that $B$ is a symmetric bilinear form on $V$. Recall that the orthogonal group of $B$ is

$$O(V) = \{g \in GL(V) \mid B(g \cdot v, g \cdot w) = B(v, w) \quad (v, w \in V)\}.$$ 

The Clifford algebra of $V$ is by definition the associative algebra $C(V)$ generated by $k$ and $V$, subject to the relations

$$v \cdot w + w \cdot v = 2B(v, w) \quad (v, w \in V).$$

You may assume the following facts about $C(V)$.

1. If $\{e_1, \ldots, e_n\}$ is a basis of $V$, then

$$\{e_A = e_{i_1}e_{i_2}\cdots e_{i_r} \mid A = \{i_1 < \cdots < i_r\} \subset \{1, \ldots, n\} \}$$

is a basis of $C(V)$. Here the empty subset corresponds to the empty product

$$e_\emptyset = 1 \in C(V).$$

(Empty products are defined to be the multiplicative identity for the same reasons that empty sums are defined to be the additive identity.) In particular, $\dim C(V) = 2^n$.

2. There is a unique linear antiautomorphism $t$ of $C(V)$ (that is, $(ab)^t = (b')(a')$) with the property that $v^t = v$ for all $v \in V$.

3. There is a unique linear automorphism $\gamma$ of $C(V)$ with the property that $\gamma(v) = -v$ for all $v \in V$. The $+1$ eigenspace of $\gamma$ is spanned by the basis elements $e_A$ with $|A|$ even, and the $-1$ eigenspace by the elements $e_B$ with $|B|$ odd.

1. Suppose that $B$ is nondegenerate. Prove that the center of $C(V)$ is spanned by 1 if $n$ is even, and has dimension 2 if $n$ is odd.

The Gram-Schmidt process shows that $V$ has a basis

$$\{e_1, \ldots, e_n\}, \quad B(e_i, e_i) = a_i \in k^\times, \quad B(e_i, e_j) = 0 \quad (i \neq j).$$

We fix such a basis, and use it to construct a basis $\{e_A\}$ of $C(V)$ as above.

Fix an index $m$, and suppose that $m \notin A = \{i_1 < \cdots < i_r\}$. Define

$$j = j(m, A), \quad 0 \leq j \leq r$$

by the requirement that $m$ follows exactly $j$ elements of $A$:

$$i_1 < \cdots < i_j < m < i_{j+1} < \cdots < i_r.$$ 

Then it is easy to calculate

$$e_m e_A = (-1)^{j(m, A)} e_{A \cup \{m\}}, \quad e_m e_{A \cup \{m\}} = (-1)^{j(m, A)} a_m e_A.$$ 

In the same way

$$e_A e_m = (-1)^{r-j(m, A)} e_{A \cup \{m\}}, \quad e_{A \cup \{m\}} e_m = (-1)^{r+1-j(m, A)} a_m e_A.$$ 

Every subset of $\{1, \ldots, n\}$ either contains $m$ or does not. Our basis of $C(V)$ therefore says that every element of $C(V)$ may be written uniquely as

$$x = \sum_{A \subseteq \{1, \ldots, m, \ldots, n\}} (c_A e_A + d_A e_{A \cup \{m\}}),$$
with $c_A$ and $d_A$ in $k$. (The symbol $\hat{m}$ means that $m$ is omitted from the list.) Using the two equations above, we can therefore compute that $e_m$ commutes with $x$ if and only if (for each $A \subset \{1, \ldots, \hat{m}, \ldots, n\}$), either $r$ is even and $d_A = 0$, or $r$ is odd and $c_A = 0$. That is, the centralizer of $e_m$ is spanned by elements $e_A$ with $|A|$ even and $m \notin A$, and $e_A$ with $|C|$ odd and $m \in A$.

If $x$ is actually in the center of $C(V)$, then it must commute with every basis element $e_m$. If $A$ is even, then $e_A$ fails to commute with $e_m$, for any $m \in A$; so the even part of the center is spanned by $e_0 = 1$. If $A$ is odd, then $e_A$ fails to commute with $e_m$ for any $m \notin A$. So the odd part of the center is zero if $n$ is even, and is spanned by $e_{\{1, \ldots, n\}}$ if $n$ is odd. (End of solution of first problem.)

Define

$$\text{Pin}(V) = \{ a \in C(V) \mid a^t a = 1, \quad \gamma(a) = \pm a, \quad \gamma(a)Va^t \subset V \}. $$

It’s very easy to see that Pin$(V)$ is a group, a subgroup of the group of invertible elements $C(V)^\times$. For every $a \in \text{Pin}(V)$, there is a linear map

$$\alpha(a) : V \to V, \quad \alpha(a)(v) = \gamma(a)va^t.$$

2. This problem is about the group Pin$(V)$ defined above.

(1) Prove that $\alpha$ is a group homomorphism.

That $\alpha(1)$ is the identity is very easy. If $a$ and $b$ belong to Pin$(V)$, then

$$\alpha(ab)(v) = \gamma(ab)vb(ab)^t \quad \text{(definition of $\alpha(ab)$)}$$

$$= \gamma(a)\gamma(b)vb(b^t)(a^t) \quad \gamma \text{ automorphism, } ^t \text{ anti-automorphism}$$

$$= \gamma(a)[\alpha(b)v]a^t \quad \text{(definition of $\alpha(b)$)}$$

$$= \alpha(a)[\alpha(b)v] \quad \text{(definition of $\alpha(a)$)}.$$

This proves that $\alpha(ab) = \alpha(a)\alpha(b)$, as we wished to show.

(2) Prove that $\alpha(a) \in O(V)$ for all $a \in \text{Pin}(V)$.

Suppose $a \in \text{Pin}(V)$, and $\gamma(a) = \epsilon a$, with $\epsilon = +1$ or $\epsilon = -1$. We know that $a^t$ is a left inverse of $a$; since $C(V)$ is finite-dimensional, a left inverse must also be a right inverse. We calculate

$$(\alpha(a)v)^2 = (\epsilon a)va^t(\epsilon a)va^t.$$

The scalar factors $\epsilon$ commute with everything, and so cancel. Then $a^ta = 1$ (part of the definition of Pin); so

$$(\alpha(a)v)^2 = av^ta^t = B(v,v)(aa^t).$$

By the remark about right inverses, the last factor is 1; so

$$B(\alpha(a)v, \alpha(a)v) = B(v,v).$$

That is, $\alpha$ preserves the quadratic form, as we wished to show.

(3) Suppose that $v \in V$ is a unit vector (that is, that $B(v,v) = 1$). Prove that $v \in \text{Pin}(V)$.

We have $\gamma(v) = -v$ (second requirement for Pin$(V)$) and $v^t = v$ by the definition of $\gamma$ and $^t$. Therefore

$$v^tv = v^2 = B(v,v) = 1$$

(first requirement for Pin$(V)$). If $B(w,v) = 0$, then by definition of the Clifford algebra $vw = -wv$, so

$$\gamma(v)uv^t = -vuv^t = wuv^t = w;$$

and

$$\gamma(v)v^t = -v^t = -v.$$
These two equations prove that $v \in \text{Pin}(V)$, and even more: that

$$\alpha(v) = s_v,$$

the reflection in the hyperplane perpendicular to $v$.

(4) **Suppose that $B$ is nondegenerate, and that every value $B(v, v)$ of the quadratic form is a perfect square in $k$.** (This happens for example if $k$ is algebraically closed, or if $k = \mathbb{R}$ and $B$ is positive definite.) **Prove that $\alpha(\text{Pin}(V)) = O(V)$.** (Hint: you may use the fact that the orthogonal group for a nondegenerate quadratic form is generated by reflections.)

If $w \in V$ has $B(w, w) \neq 0$, then by assumption $B(w, w) = c^2$ (for some scalar $c \in k^\times$), and therefore $c^{-1}w$ is a unit vector. By the previous part, $c^{-1}w$ belongs to $\text{Pin}(V)$, and

$$\alpha(c^{-1}w) = s_{c^{-1}w} = \text{reflection in hyperplane perpendicular to } c^{-1}w.$$ 

Therefore $\alpha(\text{Pin}(V))$ includes every reflection, and so (by Cartan-Dieudonné) is equal to $O(V)$.

(5) **Prove that the kernel of $\alpha$ is $\pm 1$.**

By definition $\text{Pin}(V)$ is the disjoint union of the even elements ($\gamma(a) = a$) and the odd elements ($\gamma(a) = -a$). We compute the even and odd elements of the kernel separately. Even elements $a$ act on $V$ by

$$\alpha(a)(v) = ava^t = ava^{-1};$$

so the even elements of the kernel are

$$\text{Pin}(V) \cap \text{even elements of center of } C(V).$$

We say in Problem 1 that the even elements of the center are just the scalars $k$, and

$$\text{Pin}(V) \cap k = \{z \in k \mid z^2 = 1\} = \{\pm 1\}.$$ 

The odd elements of the kernel act on $V$ by

$$\alpha(b) = -beb^t = -beb^{-1}.$$ 

Therefore the odd elements of the kernel are

$$\text{Pin}(V) \cap \text{odd elements of the anticenter } \{b \in C(V) \mid bv = -vb \ (v \in V)\}.$$ 

The anticenter can be computed in exactly the same way as the center in Problem 1; the answer is that it is nonzero only if $n$ is even, in which case it is spanned by $e_{\{1, \ldots, n\}}$. In particular, there are no nonzero odd elements in the anticenter, completing the proof that the kernel of $\alpha$ is $\pm 1$. 