These problems concern the Lie group

\[ G = SL(2, \mathbb{R}) = 2 \times 2 \text{ real matrices of determinant 1.} \]

This group has a subgroup

\[ K = SO(2) = \{ r(\theta) = \text{def} \begin{pmatrix} \cos(2\pi \theta) & \sin(2\pi \theta) \\ -\sin(2\pi \theta) & \cos(2\pi \theta) \end{pmatrix} \mid \theta \in \mathbb{R}/\mathbb{Z} \}. \]

The Lie algebra of \( G \) is

\[ g = sl(2, \mathbb{R}) = 2 \times 2 \text{ real matrices of trace 0.} \]

A basis is

\[
H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad E = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad F = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix},
\]

satisfying the relations

\[ [H, E] = 2E, \quad [H, F] = -2F, \quad [E, F] = H. \]

The three complex matrices

\[ Z = iH, \quad X = E - F, \quad Y = iE + iF \]

satisfy the bracket relations

\[ [Z, X] = 2Y, \quad [Y, Z] = 2X, \quad [X, Y] = 2Z. \]

They span the Lie algebra

\[ \mathfrak{su}(2) = 2 \times 2 \text{ complex skew-Hermitian matrices of trace 0}, \]

which is the Lie algebra of \( SU(2) \).

An \( n \)-dimensional real representation \( \pi \) of \( sl(2, \mathbb{R}) \) consists of three \( n \times n \) real matrices

\[ H_\pi, \quad E_\pi, \quad F_\pi \]

subject to the relations

\[ [H_\pi, E_\pi] = 2E_\pi, \quad [H_\pi, F_\pi] = -2F_\pi, \quad [E_\pi, F_\pi] = H_\pi. \]

Any \( n \)-dimensional real representation \( \pi \) of \( sl(2, \mathbb{R}) \) we get an \( n \)-dimensional complex representation \( \pi_c \) of \( \mathfrak{su}(2) \):

\[ Z_{\pi_c} = iH_\pi, \quad X_{\pi_c} = E_\pi - F_\pi, \quad Y_{\pi_c} = iE_\pi + iF_\pi, \]

automatically satisfying the bracket relations

\[ [Z_\pi, X_\pi] = 2Y_\pi, \quad [Y_\pi, Z_\pi] = 2X_\pi, \quad [X_\pi, Y_\pi] = 2Z_\pi. \]

(I had a minus in the middle bracket relation on the problem set; I think this version is correct.)

The subalgebra/subgroup correspondence we just proved attaches to \( \pi_c \) a group homomorphism

\[ \Pi_c: SU(2) \to GL(n, \mathbb{C}), \quad \Pi_c(\exp(aX + bY + cZ)) = \exp(aX_{\pi_c} + bY_{\pi_c} + cZ_{\pi_c}). \]

This is true because (as we proved in class a few weeks ago) \( SU(2) \) is simply connected.
1. Suppose that \( \pi \) is an \( n \)-dimensional real representation of \( \mathfrak{sl}(2, \mathbb{R}) \). Prove that the matrix \( H_\pi \) is diagonalizable with integer eigenvalues.

The big hints above provide an \( n \)-dimensional complex representation of \( SU(2) \). Part of what’s stated is that
\[
\exp(aZ_\pi) = \Pi_c \left( \exp \left( a \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \right) \right) = \Pi_c \left( \begin{pmatrix} e^{ia} & 0 \\ 0 & e^{-ia} \end{pmatrix} \right).
\]
In particular, this means that \( \exp(2\pi Z_\pi) = \Pi_c(I_2) = I_\pi \); the exponential of \( 2\pi Z_\pi \) is the identity. Now Jordan canonical form and a bit of calculation tells you that the exponential of a matrix is the identity if and only if it’s diagonalizable, and all eigenvalues are integer multiples of \( 2\pi i \). We conclude that \( Z_\pi = iH_\pi \) is diagonalizable with eigenvalues in \( i\mathbb{Z} \), and therefore \( H_\pi \) is diagonalizable with integer eigenvalues.

2. Easy linear algebra says that every element \( g \in SL(2, \mathbb{R}) \) has an expression
\[
g = r(\theta_1) \exp(tH) r(\theta_2) \quad (\theta_i \in \mathbb{R}, t \geq 0).
\]
The constant \( t \) is uniquely determined by \( g \), and depends continuously on \( g \).

(1) Prove that any Lie algebra homomorphism \( \pi: \mathfrak{sl}(2, \mathbb{R}) \to \mathfrak{gl}(n, \mathbb{R}) \) is the differential of a Lie group homomorphism \( \Pi: SL(2, \mathbb{R}) \to GL(n, \mathbb{R}) \).

(2) Prove that if
\[
\Pi: SL(2, \mathbb{R}) \to GL(n, \mathbb{R})
\]
is any Lie group homomorphism, then \( \Pi \) has closed image.

For (1), we showed in class (using Gram-Schmidt) that the inclusion of \( SU(2) \) in \( SL(2, \mathbb{C}) \) defines an isomorphism of fundamental groups, and therefore that \( \pi_1(SL(2, \mathbb{C})) \) is trivial; that is, \( SL(2, \mathbb{C}) \) is simply connected. The Lie algebra \( \mathfrak{sl}(2, \mathbb{C}) \) consists of \( 2 \times 2 \) complex matrices of trace 0; it has a complex basis \( H, E, F \) (and therefore a real basis \( (H, iH, E, iE, F, iF) \)). The six complex matrices
\[
H_\pi, iH_\pi, E_\pi, iE_\pi, F_\pi, iF_\pi
\]
therefore define a representation of \( \mathfrak{sl}(2, \mathbb{C}) \): a Lie algebra embedding
\[
\pi_C: \mathfrak{sl}(2, \mathbb{C}) \to \mathfrak{gl}(n, \mathbb{C}).
\]
Because \( SL(2, \mathbb{C}) \) is simply connected, this embedding exponentiates to
\[
\Pi_C: SL(2, \mathbb{C}) \to GL(n, \mathbb{C}).
\]
Restricting \( \Pi_C \) to the subgroup \( SL(2, \mathbb{R}) \) gives the homomorphism we want.

For (2), suppose that
\[
g(n) = r(\theta_1(n)) \cdot \exp(t(n)H) \cdot r(\theta_2(n))
\]
is a sequence in \( G \), and that \( \Pi(g(n)) \) converges to a matrix \( A \). We wish to prove that \( A \in \Pi(G) \). After passing to a subsequence, we may assume that \( r(\theta_1(n)) \) converges to some \( r(\theta_1(0)) \), and that \( r(\theta_2(n)) \) converges to \( r(\theta_2(0)) \). It follows that
\[
\lim_{n \to \infty} \exp(t(n)H_\pi) = \Pi(r(-\theta_1(0))) \cdot A \cdot \Pi(r(-\theta_2(0))).
\]
After changing basis, we may assume that \( H_\pi \) is diagonal with integer entries \( m_j \); so \( \exp(t(n)H_\pi) \) is diagonal with entries \( e^{m_j t(n)} \).

Suppose first that some \( m_j \neq 0 \). Then the assumed convergence (of \( e^{m_j t(n)} \)) forces \( t(n) \) to converge to a real number \( t(0) \), and we get
\[
A = \Pi(r(\theta_1(0)) \cdot \exp(t(0)H) \cdot r(\theta_2(0))) \in \Pi(SL(2, \mathbb{R}));
\]
so the image is closed in this case.
If all the $m_j$ are equal to zero, then we can replace $t(n)$ by 0 without changing the sequence $\Pi(g(n))$, and we get the same conclusion.

3. Of course you know that $\pi_1(SO(2)) = \mathbb{Z}$. You may assume (we did this in class) that the inclusion of $K$ defines an isomorphism

$$\pi_1(K) \simeq \pi_1(G).$$

This means that the universal cover $\tilde{G}$ of $G$ contains the universal cover $\tilde{K} = \mathbb{R}$; it’s natural to write

$$\tilde{K} = \{\tilde{r}(\theta) \mid \theta \in \mathbb{R}\}.$$

**Find an example of a Lie group $M$ with Lie algebra $m$, and an inclusion**

$$\pi: sl(2, \mathbb{R}) \rightarrowtail m$$

**so that the corresponding Lie subgroup of $M$ is not closed.**

Hint: this is hard. The simplest example I know has

$$m = sl(2, \mathbb{R}) \times sl(2, \mathbb{R}) \times \mathbb{R}.$$

Here is an example with

$$m = sl(2, \mathbb{R}) \times \mathbb{R}.$$  

The simply connected group with this Lie algebra is

$$\tilde{M} = \tilde{SL}(2, \mathbb{R}) \times \mathbb{R}.$$  

The center of a product group is the product of the centers:

$$Z(\tilde{M}) = Z(\tilde{SL}(2, \mathbb{R})) \times \mathbb{R} = \mathbb{Z} \times \mathbb{R}.$$  

Inside this center I choose the subgroup

$$\Gamma = \langle (0,1), (1,\pi) \rangle \simeq \mathbb{Z}^2;$$

this is a discrete subgroup. It’s normal in $\tilde{M}$ because it’s central. We may therefore define

$$M = \tilde{M}/\Gamma, \quad \pi_1(M) \simeq \Gamma = \mathbb{Z}^2$$

Then $M$ is a Lie group with Lie algebra $m$. The obvious inclusion

$$\pi: sl(2, \mathbb{R}) \hookrightarrow m = sl(2, \mathbb{R}) \times \mathbb{R}$$

(on the first coordinate) is the subalgebra we want. Of course the corresponding subgroup is just the image of the first factor $\tilde{SL}(2, \mathbb{R})$ of $M$. Because

$$\Gamma = \{(n, m + n\pi)m, n \in \mathbb{Z}\}$$

meets the first factor only in the identity ($m + n\pi$ can never be zero unless $m = n = 0$), the subgroup is precisely $\tilde{SL}(2, \mathbb{R})$. But I claim it is not closed. For any $\theta \in \mathbb{R}$ there is a sequence of integers $m_j$ and $n_j$ so that

$$\lim_{j \to \infty} m_j + n_j\pi = \theta.$$  

This means that the sequence of central elements $n_j \in \tilde{SL}(2, \mathbb{R})$ converges to the image of $(1, \theta)$ in $M$. That is, the closure of the embedded subgroup $\tilde{SL}(2, \mathbb{R})$ includes the central subgroup that is the image of $\mathbb{R}$. So in fact the embedded subgroup is dense in $M$. 
