18.75 fifth problems solutions

Define $V = C^\infty(\mathbb{R})$ to be the complex vector space of all infinitely differentiable complex-valued functions on the real line. The big group we will look at is $L = GL(V)$, all invertible linear transformations of the vector space of smooth functions (of a variable we’ll always call $x$).

Calculus is particularly concerned with three kinds of linear transformation:

$$(T_t f)(x) = f(x - t) \quad (t \in \mathbb{R}) \quad \text{left translation by } t$$

$$(M_\xi f)(x) = \exp(-2\pi i \xi x) f(x) \quad (\xi \in \mathbb{R}) \quad \text{multiplication by } \exp(-2\pi i \xi x)$$

$$(Z_\theta f)(x) = \exp(2\pi i \theta) f(x) \quad (\theta \in \mathbb{R}/\mathbb{Z}) \quad \text{scalar multiplication by } \exp(2\pi i \theta).$$

1. **Prove that** $T_t T_s = T_{t+s}$. **In particular (since** $T_0$ **is the identity),** $T_t$ **is invertible.**

To prove an identity about linear transformations, it’s enough to prove it after application to any vector $f \in V$. We have

$$(T_t T_s f)(x) = (T_s f)(x - t)$$

$$= f((x - t) - s)$$

$$= f(x - (t + s)) = (T_{t+s} f)(x)$$

as we wished to show.

2. **Translation by** $t$ **is a one-parameter group of diffeomorphisms of** $\mathbb{R}$. **Find the corresponding vector field** $X$ **on** $\mathbb{R}$.  

In order to make the rest of the answers come out right, I will (mis)define translation by $t$ to be $\tau_t(x) = x - t$. The definition of the corresponding vector field $X_\tau$ is

$$(X_\tau f)(x) = \frac{d}{dt} f(\tau_t(x))|_{t=0} = \lim_{s \to 0} \frac{f(\tau_s(x)) - f(\tau_0(x))}{s}$$

$$= \lim_{s \to 0} \frac{f(x - s) - f(x)}{s}$$

$$= - \frac{df}{dx}(x).$$

That is, $X_\tau = - \frac{df}{dx}$. (If you (correctly defined $\tau_t(x) = x + t$, and got $\frac{df}{dx}$, then you would need to put some signs in the next problem.)

3. **Prove that translation** $T_t$ **is characterized by a differential equation**

$$\frac{d(T_t f)}{dt} = X(T_t f).$$

(Each side of this equation is a map from $t \in \mathbb{R}$ to $V$.)

Given $f$, define

$$F(x, t) = (T_t f)(x) = f(x - t).$$

What the problem means is that the function $F$ of two variables is characterized by the differential equation

$$\frac{\partial F}{\partial t} = - \frac{\partial F}{\partial x},$$

and the initial condition

$$F(x, 0) = f(x).$$
It’s very easy to prove that (given \( f \)), the function \( F \) above satisfies the differential equation and initial condition; I’ll omit that. So assume that \( F \) satisfies the differential equation. If we change coordinates from \((x, t)\) to \((x + t, x - t)\), then the differential equation says that \( F \) is annihilated by the partial derivative with respect to the variable \( x + t \); that is, that \( F \) only depends on \( x - t \). This means (assuming the differential equation)
\[
F(x, t) = F(x', t') \quad \text{if } x - t = x' - t'.
\]
In particular, it means that
\[
F(x, t) = F(x - t, 0) = f(x - t) = (T_t f)(x)
\]
(using also the initial condition).

(4) **Prove that** \( M_\xi M_\mu = M_{\xi + \mu} \). **In particular,** \( M_\xi \) **is invertible.**
This is exactly like (1), so I’ll skip it.

(5) **Is there a differential equation like the one in (3) characterizing** \( M_\xi \)?
If we are going to imitate the argument in (3), we should introduce (given \( f(x) \)) a function of two variables
\[
\phi(x, \xi) = (M_\xi f)(x) = \exp(-2\pi i x \xi) f(x),
\]
and try to characterize it by a differential equation. A left side analogous to the one in (3) is \( \frac{\partial \phi}{\partial \xi} \); and for our \( \phi \), we calculate
\[
\frac{\partial \phi}{\partial \xi} = (-2\pi i x) \phi.
\]
This will do for a differential equation. A good initial condition is
\[
\phi(x, 0) = f(x).
\]
So our \( \phi \) clearly satisfies the differential equation and initial condition. Conversely, if \( \phi \) satisfies the differential equation, then we know from 18.03 (looking just at the function of one variable \( \xi \)) that
\[
\phi(x, \xi) = \exp(-2\pi i x \xi) \phi(x, 0).
\]
Plugging in the initial condition, we get
\[
\phi(x, \xi) = \exp(-2\pi i x \xi) f(x) = (M_\xi f)(x),
\]
as we wished to show.

(6) **You may assume that** \( Z_\theta Z_\phi = Z_{\theta + \phi} \). **Is there a differential equation characterizing** \( Z_\theta \)?
I’ll just write the result:
\[
\frac{d(Z_\theta f)}{d\theta} = 2\pi i Z_\theta f, \quad Z_0 f = f.
\]
The secret point of these differential equations is that the Lie algebra of the Heisenberg group can be identified with the three-dimensional Lie algebra spanned by the differential operators
\[
\frac{d}{dx}, \quad \text{mult by } 2\pi i x, \quad \text{mult by } 2\pi i.
\]
The last two operators have order zero, but that’s OK.
Define the Heisenberg group

\[ G = \langle T_t, M_\xi, Z_\theta \mid t \in \mathbb{R}, \xi \in \mathbb{R}, \theta \in \mathbb{R}/\mathbb{Z} \rangle \subset GL(C^\infty(\mathbb{R})). \]

Prove that every element \( g \in G \) can be written uniquely as

\[ g = Z_\theta M_\xi T_t \quad (t \in \mathbb{R}, \xi \in \mathbb{R}, \theta \in \mathbb{R}/\mathbb{Z}). \]

Define

\[ g(\theta, \xi, t) = Z_\theta M_\xi T_t. \]

This linear transformation acts on a function \( f \) by

\[ (Z_\theta M_\xi T_t f)(x) = \exp(2\pi i \theta) \exp(-2\pi i \xi x)(T_t f)(x) = \exp(2\pi i \theta) \exp(-2\pi i \xi x) f(x - t). \]

If we apply to this new function a second element \( g(\theta', \xi', t') \), we get

\[
(g(\theta', \xi', t')g(\theta, \xi, t)f)(x) = \exp(2\pi i \theta') \exp(-2\pi i \xi' x)(g(\theta, \xi, t)f)(x - t')
\]
\[
= \exp(2\pi i \theta) \exp(2\pi i \theta') \exp(-2\pi i \xi \xi' \theta) \exp(-2\pi i (x - t') \xi) f(x - t' - t)
\]
\[
= \exp(2\pi i (\theta + \theta' + t' \xi)) \exp(-2\pi i (\xi + \xi') x) f(x - t' - t)
\]
\[
= (g(\theta + \theta' + t' \xi, \xi + \xi', t + t')f)(x).
\]

Another way to write this is as a calculation of the group law:

\[ g(\theta', \xi', t')g(\theta, \xi, t) = g(\theta' + \theta + t' \xi, \xi' + \xi, t' + t). \]

From this calculation we see that the collection of all elements \( g(\theta, \xi, t) \) is closed under multiplication, includes all the elements \( Z_\theta, M_\xi, \) and \( T_t \), and includes the identity \( g(0,0,0) \).

We also calculate

\[ g(\theta, \xi, t)^{-1} = g(-\theta + t \xi, -\xi, -t). \]

Finally it’s clear that all the elements \( g(\theta, \xi, t) \) are distinct, which is the uniqueness of our decomposition.

(8) **Explain how to make** \( G \) **a Lie group. Calculate** \( \pi_1(G) \).

The coordinates given in (7) identify \( G \) with the manifold

\[ \mathbb{R}/\mathbb{Z} \times \mathbb{R} \times \mathbb{R}. \]

The multiplication law is given in these coordinates by the formula in the solution to (7), which is a smooth function of the six coordinates. Similarly the inverse is a smooth function of the coordinates. So \( G \) is a Lie group, with underlying manifold \( \mathbb{R}^2 \) times a circle. In particular, \( G \) can be contracted onto a circle; so

\[ \pi_1(G) = \pi_1(\text{circle}) = \mathbb{Z}. \]