18.755 fifth problem solutions

1. Let $V$ be the vector space $\mathcal{C}_c^\infty(\mathbb{R})$ of compactly supported smooth functions on the real line. Calculus has a lot to say about two families of linear transformations on $V$: translation by $t$

$$ (T_t f)(x) = f(x - t) \quad (t \in \mathbb{R}) $$

and multiplication by exponentials

$$ (M_\xi f)(x) = e^{ix\xi} f(x). $$

It’s very easy to check that each of these families is a group under composition of linear operators:

$$ T_t T_{t'} = T_{t+t'}, \quad T_0 = \text{identity}, $$

and similarly for $M$. In this way it’s natural to regard each of these families as a one-dimensional Lie group, isomorphic to $\mathbb{R}$. You may assume all that.

Now let $G$ be the group of linear transformations of $V$ generated by all the $T_t$ and $M_\xi$.

(1) Prove that $G$ is in a natural way a Lie group.

(2) Calculate $\pi_0(G)$ and $\pi_1(G)$.

(3) Is $G$ diffeomorphic to a group of matrices?

After experimenting with writing formulas for $T_t M_\xi$ and $M_\xi T_t$, you might be led to define another family of linear transformations

$$ (Z_\theta f)(x) = \exp(i\theta) f(x) : $$

just scalar multiplication by $\exp(i\theta)$ (and so depends only on $\theta$ modulo integer multiples of $2\pi$). Now define a linear transformation

$$ g(\theta, \xi, t) = Z_\theta M_\xi T_t. \quad \text{(COORDS)} $$

This linear transformation acts on a function $f$ by

$$ (Z_\theta M_\xi T_t f)(x) = \exp(i\theta) \exp(i\xi x)(T_t f)(x) $$

$$ = \exp(i(\theta + x\xi)) f(x - t). $$

If we apply to this new function a second element $g(\theta', \xi', t')$, we get

$$ (g(\theta', \xi', t') g(\theta, \xi, t) f)(x) = \exp(i(\theta' + x\xi')) (g(\theta, \xi, t) f)(x - t') $$

$$ = \exp(i(\theta' + x\xi')) \exp(i(\theta + (x - t')\xi)) f(x - t' - t) $$

$$ = \exp(i((\theta + \theta' - t')\xi + (x(\xi + \xi')) f(x - t' - t) $$

$$ = (g(\theta + \theta' - t'\xi, \xi + \xi', t + t') f)(x). \quad \text{(PROD)} $$
Another way to write this is as a calculation of the group law:

\[ g(\theta', \xi', t') g(\theta, \xi, t) = g(\theta' + \theta - t' \xi, \xi' + \xi, t + t'). \]

This calculation shows that the collection of all elements \( g(\theta, \xi, t) \) is closed under multiplication, includes all the elements \( Z_\theta, M_\xi, \) and \( T_t \), and includes the identity \( g(0, 0, 0) \). We also calculate

\[ g(\theta, \xi, t)^{-1} = g(-\theta - t \xi, -\xi, -t). \]

These elements therefore constitute the group \( G \). Finally it’s clear (from the formula for the action on \( f \)) that all the elements \( g(\theta, \xi, t) \) are distinct (except for adding multiples of \( 2\pi \) to \( \theta \)). The coordinates given in (COORDS) identify \( G \) with the manifold \( \mathbb{R}/2\pi \mathbb{Z} \times \mathbb{R} \times \mathbb{R} \).

The multiplication law is given in these coordinates by the formula in (PROD), which is a smooth function of the six coordinates. Similarly the inverse is a smooth function of the coordinates. So \( G \) is a Lie group, with underlying manifold \( \mathbb{R}^2 \) times a circle. This is (1).

Products of path-connected spaces are path-connected, so \( G \) is path-connected; so \( \pi_0(G) \) is trivial. Since \( \mathbb{R}^2 \) is contractible, \( G \) can be contracted onto a circle; so

\[ \pi_1(G) = \pi_1(\text{circle}) = \mathbb{Z}. \]

This is (2).

For (3), suppose we find a complex vector space \( V \) and a continuous inclusion

\[ \gamma: G \to GL(V). \]

Each one-parameter group in \( G \) must map to a one-parameter group in \( GL(n, \mathbb{C}) \); so we find linear transformations \( M, T, \) and \( Z \) so that

\[ \gamma(T_t) = \exp(tT), \quad \gamma(M_\xi) = \exp(\xi M), \quad \gamma(Z_\theta) = \exp(\theta Z). \]

Because the \( Z_\theta \) is periodic of period \( 2\pi \), the third equation forces \( Z \) to be diagonalizable with eigenvalues in \( 2\pi i \mathbb{Z} \). If we write choose a basis of eigenvectors, we find integers

\[ m_1 < m_2 < \cdots m_r, \quad d_1 > 0, \ldots, d_r > 0 \]

so that \( \gamma(Z_\theta) \) is diagonal with diagonal entries

\[ \exp(2\pi i m_1) \ (d_1 \text{ times}), \ldots, \exp(2\pi i m_r) \ (d_r \text{ times}). \]

Because the map \( \gamma \) is assumed one-to-one, the integers \( m_i \) must be relatively prime; in particular, they cannot all be zero.

Because \( T_t \) and \( M_\xi \) commute with \( Z_\theta \), it follows that \( T \) and \( M \) must commute with all these matrices \( \gamma(Z_\theta) \). Therefore \( T \) and \( M \) must be block diagonal, with blocks \( T_t \) and \( M_i \) of sizes \( d_i \) (for \( i = 1, \ldots, r \)).

The multiplication formula in \( G \) shows that

\[ M_\xi T_t M_{-\xi} T_{-t} = Z_{t \xi}. \]
Because $\gamma$ is a group homomorphism, it follows that
\[
\exp(\xi M) \exp(tT) \exp(-\xi M) \exp(-tT) = \exp(t\xi Z).
\]
All these matrices are block diagonal; so for each $j$, we get
\[
\exp(\xi M_j) \exp(tT_j) \exp(-\xi M_j) \exp(-tT_j) = \exp(2\pi t \xi m_j) I_d.
\]
The left side evidently has determinant one (determinant of any commutator is 1),
so (taking determinant of the right side), we get
\[
\exp(2\pi t \xi m_j d_j) = 1 \quad (t, \xi \in \mathbb{R}).
\]
This equation evidently forces $m_j = 0$, which contradicts our earlier discovery
that some $m_j$ must be nonzero. The conclusion is that $\gamma$ cannot exist: $G$ is not
diffeomorphic to a group of matrices.

Suppose $(M, m_0)$ is a connected manifold with a base point $m_0$, and $(G, e)$ is a
connected Lie group with (natural) base point the identity. I hope by Monday 3/9
to have defined fundamental groups and universal covering spaces; for this problem
set you can take the definitions to be
\[
\tilde{M} = \text{def} \{ \text{homotopy classes of paths in } M \text{ starting at } m_0 \},
\]
and in particular
\[
\tilde{G} = \text{def} \{ \text{homotopy classes of paths in } G \text{ starting at } e \}.
\]
A convenient notation for paths is
\[
\mu: [0, 1] \to M, \quad \mu(0) = m_0, \quad \gamma: [0, 1] \to G, \quad \gamma(0) = e.
\]
The covering maps are
\[
\pi_M: \tilde{M} \to M, \quad \pi_M(\mu) = \mu(1)
\]
and similarly for $\tilde{G}$. The group structure on $\tilde{G}$ is defined by the group multiplication
in $G$, applied pointwise to two paths.

2. With notation as above, suppose that $G$ acts (smoothly) on $M$. Explain how to define
a natural action of $\tilde{G}$ on $\tilde{M}$. Explain exactly what you need to check to see
that your definition makes sense. Write carefully some details of this checking (enough to show convincingly that you understand it).

Suppose $\gamma$ is a path in $G$ (starting at $e$) representing an element $\tilde{g} \in \tilde{G}$, and $\mu$ is
a path in $M$ (starting at $m_0$) representing an element $\tilde{m} \in \tilde{M}$. We want to define
\[
\tilde{g} \cdot \tilde{m} = \text{homotopy class of path } \gamma \cdot \mu \text{ in } M.
\]
This means the path
\[(\gamma \cdot \mu)(t) = \gamma(t) \cdot \mu(t) \quad (0 \leq t \leq 1).\]

First of all,
\[(\gamma \cdot \mu)(0) = \gamma(0) \cdot \mu(0) = e \cdot m_0 = m_0,\]
so this path starts at \(m_0\) as required. Second, the path is the composition of the continuous map
\[\gamma \times \mu: [0, 1] \to G \times M\]
with the continuous action map
\[G \times M \to M, \quad (g, m) \mapsto g \cdot m;\]
so \(\gamma \cdot \mu\) is continuous.

To see that this is well-defined, suppose \(\gamma_0\) and \(\gamma_1\) are homotopic paths in \(G\), and \(\mu_0\) and \(\mu_1\) are homotopic paths in \(M\). We must show that \(\gamma_0 \cdot \mu_0\) and \(\gamma_1 \cdot \mu_1\) are homotopic paths in \(M\). The hypothesis on \(\gamma_0\) and \(\gamma_1\) means that these two paths have a common endpoint
\[\gamma_0(1) = \gamma_1(1) = g,
\]
and that there is a continuous map
\[h: [0, 1] \times [0, 1] \to G\]
subject to
\[h(0, s) = e, \quad h(1, s) = g \quad (s \in [0, 1]),\]
\[h(t, 0) = \gamma_0(t), \quad h(t, 1) = \gamma_1(t) \quad (t \in [0, 1]).\]
The hypothesis on \(\mu_1\) and \(\mu_2\) means that these two paths have a common endpoint
\[\mu_1(1) = \mu_2(1) = m,\]
and that there is a continuous map
\[j: [0, 1] \times [0, 1] \to M\]
subject to
\[j(0, s) = m_0, \quad j(1, s) = m \quad (s \in [0, 1]),\]
\[j(t, 0) = \mu_0(t), \quad j(t, 1) = \mu_1(t) \quad (t \in [0, 1]).\]
We are required to find a homotopy
\[J: [0, 1] \times [0, 1] \to M\]
from \(\gamma_0 \cdot \mu_0\) to \(\gamma_1 \cdot \mu_1\). We can define
\[J(t, s) = h(t, s) \cdot j(t, s).\]
This is continuous because it is the composition of the continuous map

\[ h \times j : [0, 1] \times [0, 1] \to G \times M \]

with the continuous action map. Since \( h(0, s) = e \), \( h(1, s) = g \), \( j(0, s) = m_0 \), and \( j(1, s) = m \), we find

\[ J(0, s) = h(0, s) \cdot j(0, s) = e \cdot m_0 = m_0 \]

and

\[ J(1, s) = h(1, s) \cdot j(1, s) = g \cdot m, \]

as required. Similarly, we find that

\[ J(t, 0) = \gamma_0(t) \cdot \mu_0(t), \quad J(t, 1) = \gamma_1(t) \cdot \mu_1(t), \]

as required. This proves that the action is well-defined on homotopy classes of paths. Along the way we saw that in terms of the covering maps

\[ \pi_G : \tilde{G} \to G, \quad \text{class of } \gamma \mapsto \gamma(1) \]

and similarly for \( M \), we have

\[ \pi_M(\tilde{g} \cdot \tilde{m}) = \pi_G(\tilde{g}) \cdot \pi_M(\tilde{m}). \]

Because of this formula, we can say that we have “lifted” the action of \( G \) on \( M \) to an action of \( \tilde{G} \) on \( \tilde{M} \).

We should check that the well-defined map \( \tilde{G} \times \tilde{M} \to \tilde{M} \) is actually a group action. So suppose \( \gamma_1 \) and \( \gamma_2 \) are paths in \( G \) (starting at \( e \)), and \( \mu \) is a path in \( M \) (starting at \( m_0 \)). Write \([\cdot]\) for a homotopy class of paths. Then

\[ [\gamma_1] \cdot ([\gamma_2] \cdot [\mu]) = [\gamma_1] \cdot [\gamma_2 \cdot \mu] \quad \text{(def of action on } \tilde{M}) \]

\[ = [\gamma_1 \cdot (\gamma_2 \cdot \mu)] \quad \text{(def of action on } \tilde{M}) \]

\[ = [(\gamma_1 \cdot \gamma_2) \cdot \mu] \quad \text{(since } G \text{ acts on } M) \]

\[ = [\gamma_1 \cdot \gamma_2] \cdot [\mu] \quad \text{(def of action on } \tilde{M}) \]

\[ = ([\gamma_1] \cdot [\gamma_2]) \cdot [\mu] \quad \text{(def of group law in } \tilde{G}). \]

3. Suppose that \( G \) is the circle group

\[ G = \{ \exp(2\pi i \theta) \mid \theta \in \mathbb{R} \} \simeq \mathbb{R} / \mathbb{Z}. \]

You may assume that every path starting at the origin in \( G \) is homotopic to a (unique) path

\[ \gamma_\theta(t) = \exp(2\pi i t \theta) \quad (0 \leq t \leq 1), \]

so that the universal covering group is

\[ \tilde{G} = \{ \gamma_\theta \mid \theta \in \mathbb{R} \} \simeq \mathbb{R}. \]

(1) Find an action of \( G \) on a manifold \( M \) so that the action of \( \tilde{G} \) on \( \tilde{M} \) is faithful that no nontrivial element of \( \tilde{G} \) acts trivially on \( \tilde{M} \).

(2) Find a faithful action of \( G \) on a manifold \( N \) so that the action of \( \tilde{G} \) on \( \tilde{N} \) descends to \( G \); that is, that every element of \( \mathbb{Z} \subset \tilde{G} \) acts trivially on \( \tilde{N} \).

(3) Is it possible for the action of \( G \) on \( M \) in the first part not to be faithful?
For (1), you can take $M = G$, with the action of left multiplication. Then the action of $\tilde{G}$ on $\tilde{M} = \tilde{G}$ is still left multiplication, and is therefore faithful.

For (2), let $G$ act by multiplication on the disk

$$N = \{ z \in \mathbb{C} \mid |z|^2 \leq 1 \};$$

the formula is

$$\exp(2\pi i \theta) \cdot z = \exp(2\pi i \theta)z.$$ 

The only element of $G$ fixing the point $1 \in N$ is $1$, so the action is faithful. The disk is contractible, and therefore simply connected, so $\tilde{N} = N$. Because of the compatibility of the covering action with the covering maps, it follows that the covering action is

$$\gamma_\theta \cdot z = \exp(2\pi i \theta)z,$$

which of course descends to the original action of $G$ on $N$.

For (3), let $G$ act on $M_2 = G$ by the squaring map:

$$\exp(2\pi i \theta) \cdot \exp(2\pi i \phi) = \exp(2\pi i (2\theta + \phi)).$$

(This is an action because $G$ is abelian; if $G$ were not abelian, you’d get into trouble proving that you had an action because

$$g^2 \cdot h^2 \neq (gh)^2$$

whenever $g$ and $h$ don’t commute.) The action is not faithful, because the formula shows that $-1 = \exp(\pi i)$ acts trivially on $M_2$. But the universal covering action is still the squaring action

$$\gamma_\theta \cdot \gamma_\phi = \gamma_{2\theta + \phi},$$

and this is a faithful action of $\mathbb{R}$ on $\mathbb{R}$.

The point of this problem is that there is no obvious simple statement to be made about faithfulness of universal cover actions.