The first problem concerns vector fields on $\mathbb{R}$. You know that any such vector field has the form $f(x)\frac{d}{dx}$, and that the Lie bracket is

$$\left[ f \frac{d}{dx}, g \frac{d}{dx} \right] = \left( f \frac{dg}{dx} - \frac{df}{dx} g \right) \frac{d}{dx}.$$ 

I deduced in class that the Lie algebra generated by $\frac{d}{dx}$ and $g \frac{d}{dx}$ could be finite-dimensional only if the collection of derivatives of $g$

$$g, g', g'', g^{(3)}, g^{(4)}, \ldots$$

is linearly dependent; equivalently, that $g$ must satisfy an ordinary differential equation with constant coefficients. You may take it as known that the functions $g$ satisfying such a differential equation are linear combinations of functions

$$g_{m,b,a}(x) = x^m \cos(bx)e^{ax}, \quad h_{m,b,a}(x) = x^m \sin(bx)e^{ax}$$

with $a$ and $b$ real numbers and $m$ a non-negative integer.

1. Say as much as you can about the finite-dimensional Lie algebras of vector fields on $\mathbb{R}$ containing the vector field $\frac{d}{dx}$. (Best answer would be a precise classification of all of these Lie algebras. If you can’t find that, you should still be able to give some interesting examples of such Lie algebras.)

Because the Lie bracket is bilinear, for any Lie algebra $\mathfrak{g}$, the map

$$\text{Ad}(H): \mathfrak{g} \to \mathfrak{g}, \quad \text{Ad}(H)(X) = [H, X]$$

is linear. So assume that $\mathfrak{g}$ is a finite-dimensional Lie algebra of vector fields on $\mathbb{R}$ containing $\frac{d}{dx}$. I’ll do the problem by studying the linear transformation

$$D = \text{Ad}(\frac{d}{dx})$$

of $\mathfrak{g}$. Because linear algebra is easier on complex vector spaces, I’ll use the complexification

$$\mathfrak{g}_C = \mathfrak{g} \otimes_{\mathbb{R}} \mathbb{C},$$

which can be defined either formally (as symbols $X + iY$ with $X$ and $Y$ in $\mathfrak{g}$) or as complex-valued vector fields (which are defined exactly like ordinary vector fields, but using complex-valued functions instead of real-valued ones). Once we understand $\mathfrak{g}_C$, we can recover the original real vector fields as

$$\mathfrak{g} = \{ f \frac{d}{dx} \in \mathfrak{g}_C \mid f \text{ is real} \}.$$
I’ll write
\[ D_C = \text{Ad}(\frac{d}{dx}); \mathfrak{g}_C \rightarrow \mathfrak{g}_C, \]
a linear transformation of a finite-dimensional complex vector space.

List the distinct eigenvalues of \( D_C \) as
\[ \{0 = \lambda_0, \lambda_1, \ldots, \lambda_m\}. \]

The eigenvalue 0 has to occur because
\[ \text{Ad}(\frac{d}{dx})(\frac{d}{dx}) = \left[ \frac{d}{dx}, \frac{d}{dx} \right] = 0. \]

The first display in the comments before the problem shows that
\[ \left[ \frac{d}{dx}, f \frac{d}{dx} \right] = f' \frac{d}{dx}. \]

Therefore the \( \lambda \) eigenspace of \( D_C \) is
\[ \mathfrak{g}_{C,\lambda} = \{ f \frac{d}{dx} \mid f' = \lambda f \} \]
\[ = \{ Ae^{\lambda x} \frac{d}{dx} \mid A \in \mathbb{C} \}. \]

Therefore \( \mathfrak{g}_C \) must contain the \( m + 1 \) eigenvectors
\[ \frac{d}{dx}, e^{\lambda_1 x} \frac{d}{dx}, \ldots, e^{\lambda_m x} \frac{d}{dx}. \]

Using the bracket formula again, we find
\[ \left[ e^{\lambda x} \frac{d}{dx}, e^{\mu x} \frac{d}{dx} \right] = (\mu - \lambda)e^{(\lambda + \mu)x} \frac{d}{dx}. \]

This equation shows that if \( \lambda \neq \mu \) are eigenvalues, then \( \lambda + \mu \) is an eigenvalue as well. Because the set of eigenvalues is finite, we find that there are just three possibilities:

(1) A the only eigenvalue is zero; or
(2) B there is a single nonzero eigenvalue \( \lambda_1 \); or
(3) C there are exactly two nonzero eigenvalues \( \lambda_1 \) and \( -\lambda_1 = \lambda_2 \).

Each of these cases gives rise to corresponding Lie algebra of complex vector fields
\[ \mathfrak{g}_{C,A} = \mathbb{C} \frac{d}{dx}. \]
$$g_{C,B,\lambda} = \{(a + be^{\lambda x}) \frac{d}{dx} \mid a, b \in \mathbb{C}\},$$

$$g_{C,C,\lambda} = \{(a + be^{\lambda x} + ce^{-\lambda x}) \frac{d}{dx} \mid a, b, c \in \mathbb{C}\}.$$  

Extracting the real-valued coefficients among these gives for example

$$g_{R,C,\lambda} = \left\{ \begin{array}{l}
(A + B \cos(Lx) + C \sin(Lx)) \frac{d}{dx} \mid A, B, C \in \mathbb{R} \\
A \frac{d}{dx} \mid A \in \mathbb{R}
\end{array} \right\} \quad (\lambda = iL)$$

$$\quad \left\{ \begin{array}{l}
A \frac{d}{dx} \mid A \in \mathbb{R}
\end{array} \right\} \quad (\lambda \notin i\mathbb{R}).$$

Notice that many of the complex Lie algebras simply do not arise from real Lie algebras by complexification: when we complexified, we made the problem easier, but added extraneous solutions.

To continue to analyze possible Lie algebras, we need to consider the possibility that $D_C$ is not diagonalizable. By linear algebra $g_C$ is the direct sum of the generalized eigenspaces

$$g_{C,[\lambda]} = \{X \mid D_C - \lambda)^N X = 0\}.$$  

If $D_C$ is not diagonalizable on the generalized eigenspace $g_{C,[\lambda]}$, then this eigenspace must contain a vector $v$ such that $(D_C - \lambda)v$ is a nonzero eigenvector. This is a differential equation for $v$; solving it says that

$$x e^{\lambda x} \frac{d}{dx} \in g_{C,[\lambda]}.$$  

In the same way, if these two vectors do not span the generalized eigenspace, then

$$x^2 e^{\lambda x} \frac{d}{dx} \in g_{C,[\lambda]}.$$  

Computing commutators gives

$$\left[ e^{\lambda x} \frac{d}{dx}, x e^{\mu x} \frac{d}{dx} \right] = (1 + (\mu - \lambda)x)e^{(\lambda+\mu)x} \frac{d}{dx}.$$  

So if the $\mu$ eigenspace is not diagonalizable, the $\lambda$ eigenvalue occurs, and $\lambda \neq \mu$, then the $\lambda+\mu$ eigenvalue must also occur. We conclude at once that the only possible non-diagonalizable eigenspace is for $\mu = 0$.

If the $\mu = 0$ eigenspace is not diagonalizable, and the $\lambda \neq 0$ eigenspace occurs, then the formula shows that the $\lambda$ eigenspace includes $xe^{\lambda x} \frac{d}{dx}$, and is therefore not diagonalizable, a contradiction.

So we only need to analyze Lie algebras with only the zero generalized eigenspace of $D_C$. We find

$$\left[ x^a \frac{d}{dx}, x^b \frac{d}{dx} \right] = (b - a)x^{a+b-1} \frac{d}{dx}.$$  

A short argument now shows that there are just two additional Lie algebras:

$$g_{C,D} = \left\langle \frac{d}{dx}, x \frac{d}{dx} \right\rangle.$$
\[ T_{C,E} = \left\langle \frac{d}{dx}, x\frac{d}{dx}, x^2\frac{d}{dx} \right\rangle. \]

Each of these is (as written) spanned by real vector fields.

The second problem concerns the one-dimensional real projective space \( \mathbb{RP}^1 \), defined in class to consist of all straight lines through the origin in \( \mathbb{R}^2 \). I explained that the action of \( GL(2, \mathbb{R}) \) on \( \mathbb{R}^2 \) descends to an action on \( \mathbb{RP}^1 \). In particular, the rotation matrix

\[
\begin{pmatrix}
\cos \theta & \sin \theta \\
-\sin \theta & \cos \theta
\end{pmatrix}
\]

acts on \( \mathbb{RP}^1 \).

2. Write

\[
\ell(\phi) = \mathbb{R} \begin{pmatrix} \cos \phi \\ \sin \phi \end{pmatrix},
\]

a straight line through the origin in \( \mathbb{R}^2 \).

(1) Show that

\[ \mathbb{RP}^1 = \{ \ell(\phi) \mid 0 \leq \phi < \pi \}, \]

and deduce that a vector field on \( \mathbb{RP}^1 \) is \( f(\phi) \frac{d}{d\phi} \), with \( f \) a smooth function periodic of period \( \pi \).

(2) Show that

\[ r(\theta) \ell(\phi) = \ell(\phi - \theta), \]

and deduce that the action of rotations on \( \mathbb{RP}^1 \) gives rise to the vector field \(-\frac{d}{d\phi}\).

(3) Find the Lie algebra of vector fields on \( \mathbb{RP}^1 \) given by the action of \( GL(2, \mathbb{R}) \).

Comment: the two problems have some connections.