1. Suppose $A$ is an $n \times n$ real matrix. Prove that

$$\exp(A) = \lim_{N \to \infty} (I + \frac{1}{N} A)^N.$$ 

Prove also that

$$\det(\exp(A)) = \exp(\text{tr} A)$$

(with tr $A$ the sum of the diagonal entries of $A$).

Using the binomial theorem, we compute

$$(I + \frac{1}{N} A)^N = \sum_{p=0}^{N} \binom{N}{p} I^{N-p} (A/N)^p$$

$$= \sum_{p=0}^{N} \frac{N!}{p!(N-p)!} (A/N)^p$$

$$= \sum_{p=0}^{N} \frac{N(N-1) \cdots (N-p+1)}{p!} (A/N)^p$$

$$= \sum_{p=0}^{N} \left( \frac{N}{N} \frac{N-1}{N} \cdots \frac{N-p+1}{N} \right) \frac{A^p}{p!}$$

The term in square brackets in the last line is between 0 and 1, a fact that we will use repeatedly below. For a fixed value of $p$, it converges to 1 as $N$ tends to infinity.

For an $n \times n$ matrix $X$ of real numbers, define $\|X\|$ to be $n$ times the largest absolute value of an entry of $X$. Then

$$\|XY\| \leq \|X\| \|Y\|.$$ 

Conclusion is

$$\frac{1}{k!} \|B^k\| \leq \frac{\|B\|^k}{k!},$$

so the series for $\exp(B)$ converges “faster” than the power series for $\exp(\|B\|)$.

Given $A$, and $\epsilon > 0$, choose $N_1$ so large that

$$\sum_{p=N_1+1}^{\infty} \frac{\|A\|^p}{p!} < \frac{\epsilon}{2}.$$ 

Then clearly (for $N \geq N_1$)

$$\left\| \sum_{p=N_1+1}^{\infty} \left( 1 - \left[ \frac{N}{N} \frac{N-1}{N} \cdots \frac{N-p+1}{N} \right] \frac{A^p}{p!} \right) \right\| < \frac{\epsilon}{2}.$$ 

Finally, choose $N_0 \geq N_1$ so large that

$$\left( 1 - \left[ \frac{N}{N} \frac{N-1}{N} \cdots \frac{N-p+1}{N} \right] \right) \frac{\epsilon \exp(-\|A\|)}{2}$$
for $0 \leq p \leq N_1$. Then for $N \geq N_0$, we have

\[
\| \exp(A) - (I + \frac{1}{N} A)^N \| = \| \sum_{p=0}^{N_1} \left( 1 - \left[ \frac{N}{N} \cdot \frac{N-1}{N} \cdots \frac{N-p+1}{N} \right] \right) A^p \| \\
+ \sum_{p=N_1+1}^{\infty} \left( 1 - \left[ \frac{N}{N} \cdot \frac{N-1}{N} \cdots \frac{N-p+1}{N} \right] \right) \frac{A^p}{p!} \|
\leq \frac{\epsilon \exp(-\|A\|)}{2} \left[ \sum_{p=0}^{N_1} \frac{\|A\|^p}{p!} \right] + \epsilon/2
\leq \epsilon.
\]

Here we use the choice of $N_0$ to get the inequality in the first sum and the choice of $N_1$ for the second.

Using this limit formula, we deduce

\[
\det(\exp(A)) = \lim_{N \to \infty} \det(N(I + \frac{1}{N} A)).
\]

The function $\det M$ can be written as a sum of $n!$ terms, each of which is of the form

\[
\pm m_{1, \sigma(1)} \cdot m_{2, \sigma(2)} \cdots m_{n, \sigma(n)}.
\]

From this it is easy to calculate that

\[
\det(I + \frac{1}{N} A) = 1 + \frac{\text{tr} A}{N} + O(1/N^2).
\]

Here the last term means some function of $N$ that is less than or equal to $C/N^2$ in absolute value for some $C$ (depending on $A$ but not on $N$). It’s a calculus exercise that

\[
\lim_{N \to \infty} (1 + b/N + O(1/N^2))^N = \exp(b),
\]

giving the formula we want.

2. Define

\[
GL^+(n, \mathbb{R}) = \{ A \in GL(n, \mathbb{R}) \mid \det(A) > 0 \}.
\]

It follows from Problem 1 that

\[
\exp: \mathfrak{gl}(n, \mathbb{R}) \to GL^+(n, \mathbb{R}).
\]

Is this map surjective? (Remember that $\mathfrak{gl}(n, \mathbb{R})$ means all $n \times n$ real matrices.)

The answer is no for $n \geq 2$. I’ll prove this just for $n = 2$. If $B$ is any $2 \times 2$ real matrix, then the two complex eigenvalues of $B$ (the roots of the characteristic polynomial $t^2 - (\text{tr} B)t + \det B$) are either

(1) two real numbers $\beta_1$ and $\beta_2$, or

(2) two complex conjugate complex numbers $a \pm bi$ (with $b \neq 0$).
Therefore the two eigenvalues of \( \exp(B) \) are either

(1) two **positive** real numbers \( \exp(\beta_1) \) and \( \exp(\beta_2) \), or

(2) two complex conjugate complex numbers

\[
\exp(a \pm bi) = e^a \exp(\pm bi) = r \exp(\pm bi)
\]

(with \( r = e^a \)).

In the second case, the two complex numbers have the same norm.

In light of this description, we see that the matrix

\[
B = \begin{pmatrix}
-2 & 0 \\
0 & -1/2
\end{pmatrix}, \quad \det(B) = 1
\]

cannot be in the image of \( \exp \). Its two eigenvalues \(-2\) and \(-1/2\) are not positive real, so they do not fall in case (1); but they are not complex conjugates of each other, so they do not fall in case (2).

Describing the image of \( \exp \) precisely is a bit complicated. The matrix

\[
\begin{pmatrix}
-1 & 1 \\
0 & -1
\end{pmatrix}
\]

has the right eigenvalues (a complex conjugate pair) to be in the image, but in fact it is not. On the other hand, the matrix

\[
\begin{pmatrix}
-1 & 0 & 1 & 0 \\
0 & -1 & 0 & 1 \\
0 & 0 & -1 & 0 \\
0 & 0 & 0 & -1
\end{pmatrix}
\]

is in the image of \( \exp \).

3. The “standard symplectic form” on

\[ \mathbb{R}^{2n} = \{(x, y) \mid x \in \mathbb{R}^n, \ y \in \mathbb{R}^n \} \]

is

\[
\omega((x, y), (x', y')) = x \cdot y' - y \cdot x'.
\]

The **real symplectic group** is

\[
Sp(2n, \mathbb{R}) = \{g \in GL(2n, \mathbb{R}) \mid \omega(g \cdot (x, y), g \cdot (x', y')) = \omega((x, y), (x', y'))\}.
\]

**Prove that** \( Sp(2n, \mathbb{R}) \) **is a Lie group**, and calculate its dimension.

Define

\[
J_n = \begin{pmatrix}
0_n & I_n \\
-I_n & 0_n
\end{pmatrix}.
\]

Then (if we regard \( x \) and \( y \) as \( n \times 1 \) column vectors, and \((x, y)\) as a \((2n \times 1)\) column vector)

\[
\omega((x, y), (x', y')) = x \cdot y' - y \cdot x' \\
= (x, y) \cdot (y', -x') \\
= (x, y) \cdot J(x', y') \\
= (x, y)^t J(x', y').
\]
Just as I did in class for \( SO \) (I hope), this proves that
\[
Sp(2n, \mathbb{R}) = \{ g \in GL(2n, \mathbb{R}) \mid g^tJ_ng = J_n \}.
\]

So we study the smooth map
\[
\mu: 2n \times 2n \text{ matrices} \to 2n \times 2n \text{ skew-symmetric matrices}, \quad g \mapsto g^tJ_ng.
\]

It’s easy to check (just as I did for \( SO \) in class) that \( \mu \) really maps to skew-symmetric matrices, and that
\[
d\mu(X) = X^tJ_n - J_nX \quad (X \in \mathfrak{gl}(2n, \mathbb{R}))
\]
at the identity. Explicitly, this is
\[
d\mu \left( \begin{array}{cc} A & B \\ C & D \end{array} \right) = \left( \begin{array}{cc} C - C^t & D - A^t \\ -A + D^t & B^t - B \end{array} \right)
\]
(or something like that). So \( d\mu \) at the identity is a surjection from all \( 2n \times 2n \) matrices onto all skew-symmetric \( 2n \times 2n \) matrices. So \( \mu^{-1}(J_n) \) is a submanifold of \( GL(2n, \mathbb{R}) \), of dimension
\[
\dim (2n \times 2n \text{ matrices}) - \dim (2n \times 2n \text{ skew-symm matrices})
= 4n^2 - 4n(4n - 1)/2 = 2n^2 + n.
\]

4. Find a natural inclusion \( \phi: GL(n, \mathbb{R}) \hookrightarrow Sp(2n, \mathbb{R}). \)

If \( h \) is an invertible \( n \times n \) matrix, define
\[
\mu(h) = \left( \begin{array}{cc} h & 0 \\ 0 & (h^{-1})^t \end{array} \right).
\]

It’s very easy to see that \( \mu \) is an injective group homomorphism of \( GL(n, \mathbb{R}) \) into \( GL(2n, \mathbb{R}) \); we just need to show that it lands in \( Sp(2n, \mathbb{R}) \). For this, notice that (using \( (Au \cdot v = u \cdot A^tv) \))
\[
\omega(\mu(h) \cdot (x, y), \mu(h) \cdot (x', y')) = \omega((h \cdot x, (h^{-1})^t \cdot y), ((h \cdot x', (h^{-1})^t \cdot y'))
= (h \cdot x) \cdot ((h^{-1})^t \cdot y') - (h^{-1})^t \cdot y) \cdot h \cdot x'
= [(h^{-1}h) \cdot x] \cdot y' - y \cdot [(h^{-1}h) \cdot x']
= x \cdot y' - y \cdot x' = \omega((x, y), (x', y')).
\]

So \( \mu(h) \in Sp(2n, \mathbb{R}) \), as we wished to show.