1. Suppose $V$ is a vector space over a field $F$, and $U \subset V$ is a subspace. Define 

$$GL(U, V) = \{ g \in GL(V) \mid gU = U \}.$$ 

Prove that there is a short exact sequence 

$$1 \to N \to GL(U, V) \to GL(U) \times GL(V/U) \to 1,$$

and that the normal subgroup $N$ satisfies $N \simeq \text{Hom}_F(V/U, U)$ (where the group operation on the right is addition of linear maps).

If $T \in GL(U, V)$, then we can clearly define 

$$T_U : U \to U, \quad T_U(u) = T(u) \quad (u \in U)$$

$$T_{V/U} : V/U \to V/U, \quad T_{V/U}(v + U) = Tv + U \quad (v \in V).$$

That these are well-defined linear maps respecting composition is immediate. If $I$ is the identity map on $V$, then $I$ belongs to $GL(U, V)$, and by definition $I_U$ is the identity map on $U$, and $I_{V/U}$ is the identity map on $V/U$. It follows immediately that 

$$(T^{-1})_U = (T_U)^{-1}, \quad (T^{-1})_{V/U} = (T_{V/U})^{-1}.$$ 

Consequently $T_U \in GL(U)$ and $T_{V/U} \in GL(V/U)$. Therefore we can use 

$$\pi : GL(U, V) \to GL(U) \times GL(V/U), \quad \pi(T) = (T_U, T_{V/U})$$

as a map in our desired short exact sequence.

To see that $\pi$ is surjective, choose a subspace $W \subset V$ so that 

$$V = U \oplus W.$$ 

(This requires the axiom of choice if $V$ is infinite-dimensional, but I am going to sweep such issues under the rug.) We get a natural isomorphism 

$$W \simeq V/U$$

and therefore 

$$V \simeq U \oplus V/U.$$ 

This isomorphism provides an embedding 

$$GL(U) \times GL(V/U) \hookrightarrow GL(V)$$

which is a right inverse for $\pi$. It follows in particular that $\pi$ is surjective (since that is a necessary condition for admitting a right inverse).

Define $N$ to be the kernel of $\pi$. We now have the short exact sequence in the problem; what remains is to identify $N$. By definition 

$$N = \{ T \in GL(V) \mid Tu = u \quad (u \in U), \quad Tv = v + \alpha(T)(v) \quad (v \in V) \}.$$
Here \( \alpha(T)(v) \in U \); this is just the statement of what it means for \( T \) to fix the coset \( v + U \). Obviously \( \alpha(T) \) is a linear map:

\[
\alpha(T) \in \text{Hom}_F(V, U) \quad (T \in N).
\]

The first condition on \( T \) in the definition of \( N \) implies that \( \alpha(T)(U) = 0 \); so

\[
\alpha(T) \in \text{Hom}_F(V/U, U) \quad (T \in N).
\]

Conversely, if \( A \in \text{Hom}_F(V/U, U) \), then

\[
\tau(A): V \to V, \quad \tau(A)(v) = v + A(v + U)
\]

is easily seen to belong to \( N \); and

\[
\alpha(\tau(A)) = A, \quad \tau(\alpha(T)) = T \quad (T \in N, A \in \text{Hom}_F(V/U, U)).
\]

Therefore \( \alpha \) is an isomorphism of \( N \) with \( \text{Hom}_F(V/U, U) \). Here’s the calculation that it’s a group homomorphism:

\[
ST(v) = S(Tv) = S(v + \alpha(T)(v))
\]

\[
= [S(v)] + [S(\alpha(T)(v))]
\]

\[
= [v + \alpha(S)(v)] + [\alpha(T)(v) + \alpha(S)(\alpha(T)(v))]
\]

\[
= v + \alpha(S)(v) + \alpha(T)(v) = v + (\alpha(S) + \alpha(T))(v).
\]

Here to get the last equality we use the fact that \( \alpha(T)(v) \in U \), and therefore is annihilated by \( \alpha(S) \). This calculation proves that

\[
\alpha(ST) = \alpha(S) + \alpha(T) \quad (S, T \in N)
\]

as we wished to show.

A group \( S \) is called \textit{solvable} if there is a collection of subgroups

\[
\{e\} = N_0 \subset N_1 \subset \cdots \subset N_r = S
\]

so that \( N_{i-1} \text{ normal in } N_i \) (written \( N_{i-1} \triangleleft N_i \)) and \( N_i/N_{i-1} \) is \textit{abelian} \((1 \leq i \leq r)\).

Subgroups \( H_1 \) and \( H_2 \) are \textit{conjugate} if there is \( g \in G \) such that \( gH_1g^{-1} = H_2 \).

2. Suppose \( n \geq 1 \) is an integer. Define \( G = GL(n, \mathbb{C}) \) to be the group of all \( n \times n \) invertible complex matrices, and

\[
B = \{g = (g_{ij}) \in G \mid i > j \implies g_{ij} = 0\}
\]

the subgroup of upper triangular matrices.

(1) Prove that \( B \) is solvable.

(2) Prove or give a counterexample: every element \( g \in G \) is conjugate to an element of \( B \).

(3) Prove or give a counterexample: if \( S \) is a solvable subgroup of \( G \), then \( S \) is conjugate to a subgroup of \( B \).
The subgroup $B$ is related to the chain of subspaces of $\mathbb{C}^n$

$$V_0 = 0 \subset V_1 = \mathbb{C}^1 \subset \cdots \subset V_n = \mathbb{C}^n;$$

$V_i$ is the vectors for which the last $n-i$ coordinates are zero. If we write $(e_1, \ldots, e_n)$ for the standard basis of $\mathbb{C}^n$, then

$$V_i = \text{span}(e_1, \ldots, e_i).$$

Because the $i$th column of a matrix $g$ is $g \cdot e_i$, we see that the definition of $B$ amounts to

$$B = \{ g \in GL(n, \mathbb{C}) \mid g \cdot e_i \in \text{span}(e_1, \ldots, e_i),$$

and therefore that

$$B = \{ g \in GL(n, \mathbb{C}) \mid g \cdot V_i = V_i \}. $$

Now we use some of the maps from Problem 1, and define subgroups

$$N_n = B, \quad N_j = \{ g \in B \mid (g-I)V_i \subset V_{i-(n-j)} \quad (j \leq i \leq n) \}. $$

Here for $0 \leq j < n$, $N_j$ consists of upper-triangular matrices with 1s on the diagonal and 0s on the next $n-j-1$ lines above the diagonal. This description shows clearly that

$$\{1\} = N_0 \subset N_1 \subset \cdots \subset N_{n-1} \subset N_n = B.$$

Here is the case $n = 3$:

$$N_3 = \left\{ \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & 0 & a_{33} \end{pmatrix} \mid a_{ii} \neq 0 \right\}$$

$$N_2 = \left\{ \begin{pmatrix} 1 & a_{12} & a_{13} \\ 0 & 1 & a_{23} \\ 0 & 0 & 1 \end{pmatrix} \right\}$$

$$N_1 = \left\{ \begin{pmatrix} 1 & 0 & a_{13} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right\}$$

$$N_0 = \left\{ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right\}$$

Because $B$ preserves each $V_i$, it is also clear that $N_{j-1} \triangleleft B$, and consequently that $N_{j-1} \triangleleft N_j$. So $N_j/N_{j-1}$ is a group; we just need to see that it’s abelian.

We know that $B|_{V_j} \subset GL(V_{j-1}, V_j)$, so Problem 1 gives a group homomorphism

$$N_0 = B \to \prod_{j=1}^{n} GL(V_j/V_{j-1}) = \prod_{j=1}^{n} \mathbb{C}^\times;$$

the $j$th coordinate is the $j$th diagonal entry of a matrix in $B$. This description of the map shows that it is onto, with kernel precisely $N_{n-1}$; so $N_n/N_{n-1}$ is abelian (a product of $n$ copies of $\mathbb{C}^\times$).
In the same way, for \( j \geq 2 \),

\[
B|_{V_j} \subset GL(V_{j-2}, V_j) \to GL(V_j/V_{j-2}) \simeq GL(2, \mathbb{C});
\]
the image is by definition contained in \( GL(V_j/V_{j-2}, V_{j-1}/V_{j-2}) \), the group of \( 2 \times 2 \) upper-triangular matrices. The subgroup \( N_{n-1} \) by definition maps to the kernel of

\[
GL(V_j/V_{j-2}, V_{j-1}/V_{j-2}) \to GL(V_j/V_{j-1}) \times GL(V_{j-1}/V_{j-2}),
\]
which we know from Problem 1 is

\[
\text{Hom}_C(V_j/V_{j-1}, V_{j-1}/V_{j-2}) \simeq \mathbb{C}.
\]

Adding these maps, we get

\[
N_{n-1} \to \prod_{j=2}^n \text{Hom}_C(V_j/V_{j-1}, V_{j-1}/V_{j-2}) = \prod_{j=2}^n \mathbb{C};
\]
the \( j \)th coordinate is the \((j-1, j)\) entry of a matrix in \( N_{n-1} \). This description of the map shows that it is onto, with kernel precisely \( N_{n-2} \); so \( N_{n-1}/N_{n-2} \) is abelian (a product of \( n-1 \) copies of \( \mathbb{C} \)).

A precisely parallel argument shows that for any \( i, n-1 \geq i \geq 1 \),

\[
N_i/N_{i-1} \simeq \mathbb{C}^i,
\]
an abelian group. This proves (1).

For (2), the statement is true; you learn in linear algebra that any \( n \times n \) complex matrix is conjugate to an upper triangular matrix.

For (3), the statement is false. Suppose \( n = 2 \), and let \( S \) be the group of order 8 generated by the two matrices

\[
A = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.
\]

This is the quaternion group of order eight; \( A \) corresponds to the element usually called \( i \) and \( B \) to the matrix usually called \( j \), and they satisfy the defining relations

\[
A^4 = B^4 = I, \quad ABA^{-1} = B^{-1} \neq B.
\]

To say that \( S \) is conjugate to a subgroup of \( B \) is to say that there is a chain of subspaces \( W_0 \subset W_1 \subset W_2 \) with \( \text{dim} W_i = i \) and

\[
AW_i = W_i, \quad BW_i = W_i.
\]

Of course we can and must take \( W_0 = \{0\} \), \( W_2 = \mathbb{C}^2 \); the only problem is finding the line \( W_1 \) that is an eigenspace for each of \( A \) and \( B \). The only eigenspaces for \( A \) are the coordinate axes, and neither is an eigenspace for \( B \). So there is no \( W_1 \).

3. Let \( G = SL(2, \mathbb{R}) \), the group of \( 2 \times 2 \) real matrices of determinant 1.
   (1) Prove that the subgroups

\[
H_1 = \left\{ \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} \mid t \in \mathbb{R} \right\}, \quad H_2 = \left\{ \begin{pmatrix} 1 & 0 \\ s & 1 \end{pmatrix} \mid s \in \mathbb{R} \right\}
\]

are conjugate.

(2) Find as many non-conjugate connected subgroups \( H \subset G \) as you can. You should prove that your subgroups are not conjugate.
For (1), let \( g = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \). Then \( gH_1g^{-1} = H_2 \).

For (2), here are examples:

1. \( S_1 = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right\} \), the trivial subgroup.

2. \( S_2 = \left\{ \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} \mid t \in \mathbb{R} \right\} \), a commutative group of unipotent matrices (all eigenvalues equal to 1).

3. \( S_3 = \left\{ \begin{pmatrix} s & 0 \\ 0 & s^{-1} \end{pmatrix} \mid 0 < s \in \mathbb{R} \right\} \), a commutative group of hyperbolic matrices (diagonalizable with real eigenvalues).

4. \( S_4 = \left\{ \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix} \mid \theta \in \mathbb{R} \right\} \), a commutative group of elliptic matrices (complex eigenvalues of absolute value 1).

5. \( S_5 = \left\{ \begin{pmatrix} s & t \\ 0 & s^{-1} \end{pmatrix} \mid t \in \mathbb{R}, \ 0 < s \in \mathbb{R} \right\} \), a noncommutative proper subgroup of \( G \).

6. \( S_6 = G \).

The descriptors given for each subgroup are unchanged by conjugation; this gives proofs that the subgroups are not conjugate. Connectedness is easy for cases 1)–5) (where the subgroup is written as a continuous image of a connected set like \( \mathbb{R} \) or \( \mathbb{R}^+ \)). That \( G \) itself is connected is not quite obvious; you might think about how to prove that.

These are in fact all the connected subgroups of \( G \) up to conjugacy. We don’t yet have tools to prove that, but soon.

How could you have thought of these examples? Well, if you take \( \mathbb{R} = U \subset V = \mathbb{R}^2 \), then (in the notation of Problem 1)

\[
GL(U, V) = \left\{ \begin{pmatrix} s_1 & t \\ 0 & s_2 \end{pmatrix} \mid 0 \neq s_i \in \mathbb{R}, \ t \in \mathbb{R} \right\}.
\]

The intersection with \( SL(2, \mathbb{R}) \) is

\[
SL(U, V) = \left\{ \begin{pmatrix} s & t \\ 0 & s^{-1} \end{pmatrix} \mid 0 \neq s \in \mathbb{R}, \ t \in \mathbb{R} \right\}.
\]

This group is homeomorphic to \( \mathbb{R}^+ \times \mathbb{R} \) and is therefore not connected. In a topological group \( H \), the smallest connected set containing the identity element is always a subgroup (called the identity component of \( H \)); so from \( SL(U, V) \) we get \( S_5 \).

The group \( N \) of Problem 1 is \( S_2 \), and \( S_3 \) is the identity component of \( GL(U) \times GL(W) \) (appearing in the solution of Problem 1). Hard to imagine how to invent \( S_4 \), but perhaps you’ve seen it before.