18.755 twelfth and last problems, due in class Tuesday, December 5, 2017

This problem set is partly about coverings of compact connected Lie groups, and therefore about coverings of root data. Here is some background about tori. Recall that for us a torus is a compact connected abelian Lie group $T$, and a lattice is a finitely-generated torsion-free abelian group. There is a covariant equivalence of categories

$$(\text{tori}) \to (\text{lattices}), \quad T \mapsto X_*(T) = \text{def} \, \text{Hom}(S^1, T).$$

There is a contravariant equivalence of categories

$$(\text{tori}) \to (\text{lattices}), \quad T \mapsto X^*(T) = \text{def} \, \text{Hom}(T, S^1).$$

There is a contravariant equivalence of categories

$$(\text{lattices}) \to (\text{lattices}), \quad X \mapsto X^* = \text{def} \, \text{Hom}(X, \mathbb{Z}).$$

Indeed there is a natural isomorphism

$$X^*(T) \simeq [X_*(T)]^*$$

which we use as an excuse for the otherwise confusing and ambiguous notation.

A finite cover of a torus $T$ is a short exact sequence

$$1 \to F \to \tilde{T} \to T \to 1$$

with $\tilde{T}$ another torus and $F$ a finite subgroup of $\tilde{T}$. (So the cover includes the group $\tilde{T}$, its finite subgroup $F$, and the isomorphism $\tilde{T}/F \simeq T$.) What does this look like in the world of lattices? The covering map $\pi$ induces by the category equivalence a map

$$X_*(\pi): X_*(\tilde{T}) \to X_*(T).$$

This map is one-to-one but need not be onto. Similarly, we get

$$X^*(\pi): X^*(T) \to X^*(\tilde{T}),$$

also one-to-one but not necessarily onto. We define a covering of a lattice $X$ to be a sublattice $X' \subset X$ of finite index. Similarly, a co-covering of a lattice $Y$ is a lattice $Y'$ with an inclusion $Y \subset Y'$ of finite index.

Any lattice covering $X' \hookrightarrow X$ of $X$ induces a co-covering $X^* \hookrightarrow (X')^*$ of $X^*$ by the functoriality of $*$ on lattices.
**Proposition.** The equivalence of categories $X_*$ identifies coverings $(F, \tilde{T})$ of $T$ with coverings $X' \subset X_*(T)$. In this identification,

$$F \simeq \text{Tor}(X_*(T)/X', \mathbb{Q}/\mathbb{Z}) \simeq X_*(T)/X'.$$

Similarly, the equivalence of categories $X^*$ identifies coverings $(F, \tilde{T})$ of $T$ with co-coverings $Y' \subset X^*(T)$. In this identification,

$$\tilde{F} = \text{Hom}(F, S^1) \simeq \text{Hom}(F, \mathbb{Q}/\mathbb{Z}) \simeq Y'/X^*(T)$$

$$F \simeq \text{Hom}(\tilde{F}, S^1) \simeq \text{Hom}(\tilde{F}, \mathbb{Q}/\mathbb{Z}) \simeq \text{Ext}(Y'/X^*(T), \mathbb{Z}).$$

These two identifications are related by the duality functor between coverings and co-coverings.

We can run all the same ideas backwards to study finite quotients (precisely, quotients by finite subgroups)

$$T = T/F$$

A *quotient* of a lattice $X$ is by definition a lattice $X'$ containing $X$ as a sublattice of finite index. A *co-quotient* of a lattice $Y$ is a sublattice $Y' \subset Y$ of finite index. If $X'$ is a quotient of $X$, then $(X')^*$ is a co-quotient of $X^*$. I won’t write the analogue of the proposition above relating quotients of $T$ to quotients of $X_*(T)$ and co-quotients of $X^*(T)$.

I won’t write again the definition of a root datum

$$\mathcal{R} = (X_*, R^\vee, X^*, R),$$

except to recall that $X_*$ and $X^*$ are required to be dual lattices. A *covering* of $\mathcal{R}$ is a covering $X' \subset X_*$ with the property that

$$\tilde{\mathcal{R}} = (X', R^\vee, (X')^*, R)$$

is still a root datum. The only nontrivial requirement is

$R^\vee$ is contained in the sublattice $X' \subset X_*$.

Similarly, a *quotient* of $\mathcal{R}$ is a quotient $Y' \subset X^*$ with the property that

$$\overline{\mathcal{R}} = ((Y')^*, R^\vee, Y', R)$$

is still a root datum. The only nontrivial requirement is

$R$ is contained in the sublattice $Y \subset X^*$.

A *finite cover* of a compact connected Lie group $G$ is a short exact sequence

$$1 \rightarrow F \rightarrow \tilde{G} \xrightarrow{\pi} G \rightarrow 1$$
with \( \widetilde{G} \) another compact connected Lie group and \( F \) a finite central subgroup of \( \widetilde{G} \). A finite quotient of \( G \) is a short exact sequence

\[
1 \to F \to G \xrightarrow{\pi} \widetilde{G} \to 1
\]

with \( F \) a finite central subgroup of \( G \). I am explaining in class that compact connected Lie groups (up to isomorphism) are the same as root data. You may assume that finite coverings (respectively finite quotients) of compact connected Lie groups are the same as coverings (respectively quotients) of their root data.

1. Fix an integer \( n \geq 2 \), and consider the two lattices of rank \( n \)

\[
X = \{ x \in \mathbb{Z}^n \mid \sum_{i=1}^{n} x_i \in 2\mathbb{Z} \}
\]

\[
Y = \mathbb{Z}^n \cup (\mathbb{Z} + 1/2)^n.
\]

We saw in class that these lattices are dual to each other. You may assume that the coroots and roots of \( \text{Spin}(2n) \) are

\[
R^\vee = \{ \pm e_i \pm e_j \mid 1 \leq i \neq j \leq 2m \} \subset X,
\]

\[
R = \{ \pm e_i \pm e_j \mid 1 \leq i \neq j \leq 2m \} \subset Y.
\]

Write

\[
\mathcal{R} = (X, R^\vee, Y, R)
\]

for the root datum of \( \text{Spin}(2n) \).

(1) Prove that \( \mathcal{R} \) has no coverings except itself.

(2) Find all the quotients of \( \mathcal{R} \). (Hint: the number of possibilities is odd, and it isn’t one.)

(3) I talked in class about the operation of “Langlands duality” on root data, interchanging the lattice with the dual lattice (and roots with coroots). What is the operation of Langlands duality on the various quotients of \( \mathcal{R} \)?

2. For this problem we again fix an integer \( n \geq 2 \), and consider the two lattices of rank \( n - 1 \)

\[
X = \{ x \in \mathbb{Z}^n \mid \sum x_i = 0 \},
\]

\[
Y = \mathbb{Z}^n / [\mathbb{Z}(1, \ldots, 1)].
\]

We computed in class that these are the lattices for the root datum of \( SU(n) \), with roots and coroots

\[
R^\vee = \{ e_i - e_j \mid 1 \leq i \neq j \leq n \} \subset X,
\]

\[
R = \text{images of } \{ e_i - e_j \mid 1 \leq i \neq j \leq n \} \subset Y.
\]

Answer the same questions as for \( \text{Spin}(n) \) above. (One change is that the hint is no longer correct. What is the number of finite quotients of \( SU(n) \)?)