18.755 eleventh problems, due in class Tuesday, November 21, 2017

1. Find all root data living on the lattices $X_\ast = \mathbb{Z}$, $X^\ast = \mathbb{Z}$.

2. Find all root data living on the lattices $X_\ast = \mathbb{Z}^2$, $X^\ast = \mathbb{Z}^2$ with the property that $R$ contains the two roots
   $\alpha = (1,0), \quad \beta = (0,1)$.

3. Let
   \[ X_\ast = \{ (x_1, x_2, x_3, x_4) \in \mathbb{Z}^4 \mid x_1 + x_2 + x_3 + x_4 \in 2\mathbb{Z} \}, \]
   a lattice of rank four. You may assume that the dual lattice is
   \[ X^\ast = \{ (\lambda_1, \lambda_2, \lambda_3, \lambda_4) \mid \text{(all $\lambda_j \in \mathbb{Z}$) or (all $\lambda_j \in \mathbb{Z} + 1/2$)} \}. \]
   There is a root datum living on these lattices with
   \[ R^\vee_0 = \{ \pm e_i \pm e_j \mid 1 \leq i \neq j \leq 4 \} = R_0, \]
   the bijection between roots and coroots being the “obvious” one. You may assume that
   \[ (X_\ast, R^\vee_0, X^\ast, R_0) \]
   is actually a root datum. Find all root data living on $X_\ast$ and $X^\ast$ with the property that
   \[ R^\vee_0 \subset R^\vee, \quad R_0 \subset R. \]
   (Hint: there are more than two.)