18.755 tenth problem solutions

This problem set concerns the notion of root datum, which I'll define in class probably early in the week of April 20. To do the problem set, you should look at the notes on root systems [http://www-math.mit.edu/~dav/roots.pdf](http://www-math.mit.edu/~dav/roots.pdf) linked from the class web site. Root datum is defined in Definition 1.6; just looking at Definitions 1.3, 1.6, and 4.1 should be enough to do the problems.

1. Find all root data living on the lattices \( X_* = \mathbb{Z}, X^* = \mathbb{Z} \).

Need to find what [root,coroot] pairs are to be included in the root datum. A candidate for such a pair is 
\[
\alpha = a \in \mathbb{Z}, \quad \alpha^\vee = x \in \mathbb{Z}.
\]
Axiom RD1.5 from the notes says \( ax = 2 \). There are four solutions:
\[
[\alpha, \alpha^\vee] = (2, 1) \text{ or } (-2, -1) \text{ or } (1, 2) \text{ or } (-1, -2).
\]
For any of these four pairs we calculate (using Definition 1.3 in the notes)
\[
s_{\alpha^\vee, \alpha}(\ell) = -\ell:
\]
for example,
\[
s_{1, 2}(\ell) = \ell - < \ell, 1 > \cdot 2 = \ell - 2 \ell = -\ell.
\]
So RD3 says that if \((\alpha, \alpha^\vee)\) belongs to the root datum, so does \((-\alpha, -\alpha^\vee)\). The conclusion is that there are exactly four possible root data:
\[
R = \emptyset, R^\vee = \emptyset; \quad R = \{\pm 2\}, R^\vee = \{\pm 1\};
\]
\[
R = \{\pm 1\}, R^\vee = \{\pm 2\}; \quad R = \{\pm 1, \pm 2\}, R^\vee = \{\pm 2, \pm 1\}.
\]
These root data are known as \(T_1\), \(C_1\), \(B_1\), and \(BC_1\) respectively.

2. Find all root data living on the lattices \( X_* = \mathbb{Z}^2, X^* = \mathbb{Z}^2 \) containing the two [root,coroot] pairs
\[
[\alpha, \alpha^\vee] = [(1, 0), (2, 0)] \quad [\beta, \beta^\vee] = [(0, 1), (0, 2)].
\]

The reason this problem is here is to help you understand or discover for yourself the ideas in the general classification of root systems. A key step in that classification is the case of rank two. There in turn the key fact is that any product of two reflections must again be a matrix of finite order. That’s what I’ve used in this solution.

For the given pairs, the reflections are
\[
s_{\alpha, \alpha^\vee}(a, b) = (-a, b), \quad s_{\beta, \beta^\vee}(a, b) = (a, -b)
\]
and similarly for $s_{\alpha^\vee,\alpha}$ and so on. (We’ll write $s_\alpha$ to simplify.) Each additional [root,coroot] pair looks like

$$\gamma, \gamma^\vee = [(a, b), (x, y)], \quad ax + by = 2.$$  

Such a pair must be present (according to RD3) with all of the pairs coming from the action of $s_\alpha$ and $s_\beta$:

$$[(\epsilon a, \delta b), (\epsilon x, \delta y)] \quad (\epsilon = \pm 1, \quad \delta = \pm 1).$$

So we can study first the case $a \geq 0, b \geq 0$. Using the formula from Definition 1.3 of the notes for the action of the reflection $s_\gamma$ on $X^*$, we get

$$s_\alpha = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \quad s_\beta = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad s_\gamma = \begin{pmatrix} 1 - ax & -ay \\ -bx & 1 - by \end{pmatrix}.$$  

The Weyl group $W$ includes the group generated by these three matrices. By RD3, $W$ must permute the finite set of roots, so $W$ maps to a finite permutation group. Since the roots include a basis of $X^*$, this map of $W$ must be one-to-one; so $W$ itself is finite, so every element of $W$ has finite order; so the complex eigenvalues of every element of $W$ must be roots of 1. So every matrix in $W$ must have trace between $-2$ and 2. Applying this to $s_\gamma s_\alpha$ gives

$$-2 \leq ax - by \leq 2.$$  

Adding the requirement $ax + by = 2$ gives

$$0 \leq ax \leq 2.$$  

Similarly

$$0 \leq by \leq 2.$$  

First consider the case $ax = 2, b = y = 0$. This means that the new [root,coroot] pair (recalling that we have arranged $a$ nonnegative) must be

$$[2\alpha, \alpha^\vee/2] = [(2, 0), (1, 0)].$$  

Next consider the case $ax = 2, by = 0, b \neq 0$. The reflection formulas above become

$$s_\alpha = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \quad s_\beta = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad s_\gamma = \begin{pmatrix} 1 - ax & -ay \\ -bx & 1 - by \end{pmatrix}.$$  

Then

$$s_\gamma s_\alpha = \begin{pmatrix} 1 & 0 \\ xb & 1 \end{pmatrix}.$$  

Since $x$ and $b$ are nonzero, this matrix has infinite order, so the case cannot arise. Similarly one can dispose of $by = 0, y \neq 0$; and also the cases $by = 2, a$ and $x$ not both zero.

In the remaining case $ax = by = 1$, the [root,coroot] pair we add is

$$[\gamma_{++}, \gamma^\vee_{++}] = [(1, 1), (1, 1)].$$
Because of the action of $s_\alpha$ and $s_\beta$, if we add this pair we must add all four pairs 
\[ [\gamma_\epsilon \delta, \gamma_\epsilon \delta^\vee], \quad \epsilon = \pm, \quad \delta = \pm. \]

Here is a complete list of the possible root data.

\[
\begin{align*}
R &= \{\pm \alpha, \pm \beta\}, \quad R^\vee = \{\pm \alpha^\vee, \pm \beta^\vee\} & B_1.B_1 \\
R &= \{\pm \alpha, \pm 2\alpha, \pm \beta\}, \quad R^\vee = \{\pm \alpha^\vee, \pm \alpha^\vee/2, \pm \beta^\vee\} & BC_1.B_1 \\
R &= \{\pm \alpha, \pm \beta, \pm 2\beta\}, \quad R^\vee = \{\pm \alpha^\vee, \pm \beta^\vee, \pm \beta^\vee/2\} & B_1.BC_1 \\
R &= \{\pm \alpha, \pm 2\alpha, \pm \beta, \pm 2\beta\}, \quad R^\vee = \{\pm \alpha^\vee, \pm \alpha^\vee/2, \pm \beta^\vee, \pm \beta^\vee/2\} & BC_1.BC_1 \\
R &= \{\pm \alpha, \pm \beta, \gamma_\epsilon \delta\}, \quad R^\vee = \{\pm \alpha^\vee, \pm \beta^\vee, \gamma_\epsilon ^\vee \delta^\vee\} & B_2 \\
R &= \{\pm \alpha, \pm 2\alpha, \pm \beta, \pm 2\beta, \gamma_\epsilon \delta\}, \quad R^\vee = \{\pm \alpha^\vee, \pm \alpha^\vee/2, \pm \beta^\vee, \pm \beta^\vee/2, \gamma_\epsilon ^\vee \delta^\vee\} & BC_2
\end{align*}
\]

In the last two formulas as usual $\epsilon = \pm, \delta = \pm$.

3. For each example in Problem 2, calculate the Weyl group (notes, Definition 4.1).

This is the group generated by $s_\alpha$, $s_\beta$, and perhaps $s_\gamma_\epsilon \delta$. We wrote simple formulas for the first two in Problem 2, and the third (plugging in $a = x = \epsilon, b = y = \delta$) is

\[
s_\gamma_\epsilon \delta = \begin{pmatrix} 0 & -\epsilon \delta \\ -\epsilon \delta & 0 \end{pmatrix}.
\]

This matrix acts by interchanging the two coordinates, and (if $\epsilon \neq \delta$) changing the signs of both.

It follows at once that the first four root data all have Weyl group $\mathbb{Z}_2 \times \mathbb{Z}_2$, acting by changing the signs of one or both coordinates. The last two have Weyl group the dihedral group of order 8, acting by permutation and sign changes of the two coordinates.