1. Suppose $V$ is a vector space over a field $F$, and $U \subset V$ is a subspace. Define

$$GL(U, V) = \{ g \in GL(V) \mid gU = U \}.$$ 

Prove that there is a short exact sequence

$$1 \to N \to GL(U, V) \to GL(U) \times GL(V/U) \to 1,$$

and that the normal subgroup $N$ satisfies $N \simeq \text{Hom}_F(V/U, U)$ (where the group operation on the right is addition of linear maps).

A group $S$ is called solvable if there is a collection of subgroups

$$\{e\} = N_0 \subset N_1 \subset \cdots \subset N_r = S$$

so that $N_{i-1}$ normal in $N_i$ (written $N_{i-1} \triangleleft N_i$) and $N_i/N_{i-1}$ is abelian ($1 \leq i \leq r$).

Subgroups $H_1$ and $H_2$ are conjugate if there is $g \in G$ such that $gH_1g^{-1} = H_2$.

2. Suppose $n \geq 1$ is an integer. Define $G = GL(n, \mathbb{C})$ to be the group of all $n \times n$ invertible complex matrices, and

$$B = \{ g = (g_{ij}) \in G \mid i > j \implies g_{ij} = 0 \}$$

the subgroup of upper triangular matrices.

(1) Prove that $B$ is solvable.

(2) Prove or give a counterexample: every element $g \in G$ is conjugate to an element of $B$.

(3) Prove or give a counterexample: if $S$ is a solvable subgroup of $G$, then $S$ is conjugate to a subgroup of $B$.

3. Let $G = SL(2, \mathbb{R})$, the group of $2 \times 2$ real matrices of determinant 1.

(1) Prove that the subgroups

$$H_1 = \left\{ \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} \mid t \in \mathbb{R} \right\}, \quad H_2 = \left\{ \begin{pmatrix} 1 & 0 \\ s & 1 \end{pmatrix} \mid s \in \mathbb{R} \right\}$$

are conjugate.

(2) Find as many non-conjugate connected subgroups $H \subset G$ as you can. You should prove that your subgroups are not conjugate.