1. Suppose $V$ is a vector space over a field $F$, and $U \subset V$ is a subspace. Define

$$GL(U, V) = \{g \in GL(V) \mid gU = U\}.$$ 

Prove that there is a short exact sequence

$$1 \rightarrow N \rightarrow GL(U, V) \rightarrow GL(U) \times GL(V/U) \rightarrow 1,$$

and that the normal subgroup $N$ satisfies $N \cong \text{Hom}_F(V/U, U)$ (where the group operation on the right is addition of linear maps).

Two subgroups $H_1$ and $H_2$ of $G$ are called conjugate if there is an element $g \in G$ such that $gH_1g^{-1} = H_2$.

2. Let $S_n$ be the symmetric group of all permutations of $\{1, \ldots, n\}$. For any partition of $n$

$$\pi = (p_1, \ldots, p_r), \quad p_1 \geq p_2 \geq \cdots \geq p_r > 0, \quad p_1 + \cdots + p_r = n,$$

define

$$S_\pi = \{g \in S_n \mid g(\{1, \ldots, p_1\}) = \{1, \ldots, p_1\},
\quad g(\{p_1 + 1, \ldots, p_1 + p_2\}) = \{p_1 + 1, \ldots, p_1 + p_2\}, \ldots,
\quad g(\{p_1 + \cdots + p_{r-1} + 1, \ldots, n\}) = \{p_1 + \cdots + p_{r-1} + 1, \ldots, n\}\}.\$$

Prove that if $\pi \neq \pi'$ are partitions of $n$, then $S_\pi$ is not conjugate to $S_{\pi'}$.

3. Let $G = SL(2, \mathbb{R})$, the group of $2 \times 2$ real matrices of determinant 1.

(1) Prove that the subgroups

$$H_1 = \left\{ \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} \mid t \in \mathbb{R} \right\}, \quad H_2 = \left\{ \begin{pmatrix} 1 & 0 \\ s & 1 \end{pmatrix} \mid s \in \mathbb{R} \right\}$$

are conjugate.

(2) Find as many non-conjugate connected subgroups $H \subset G$ as you can. You should prove that your subgroups are not conjugate.