1. Describe all possible symmetric bilinear forms on \( \mathbb{R}^n \) up to the action of \( GL(n, \mathbb{R}) \). That is, describe the orbits of \( GL(n, \mathbb{R}) \) on symmetric \( n \times n \) matrices. (Hint: there are \((n+1)(n+2)/2\) orbits.)

This is the classification of quadratic forms over the real numbers, which is *Sylvester's law of inertia*; one can find proofs in many places. Here is a statement.

**Theorem.** Suppose \( V \) is an \( n \)-dimensional real vector space, and \( \beta \) is a symmetric bilinear form on \( V \). Then there is an orthogonal basis

\[
(e_q, \ldots, e_p, f_1, \ldots, f_q, g_1, \ldots, g_r)
\]

for \( V \) (so that \( p + q + r = n \) with the property that

\[
\beta(e_i, e_i) = 1, \quad \beta(f_j, f_j) = -1, \quad \beta(g_k, g_k) = 0.
\]

The integer \( p \) is equal to the dimension of any maximal subspace of \( V \) on which \( \beta \) is positive definite, and is therefore uniquely determined by \( \beta \). Similarly, the integer \( q \) is equal to the dimension of any maximal subspace of \( V \) on which \( \beta \) is negative definite, and so is uniquely determined by \( \beta \). The integer \( r \) is the dimension of the radical of \( \beta \), and so is uniquely determined by \( \beta \).

Because change of basis is exactly the action of \( GL(V) \), this theorem implies

**Corollary.** The set of \( GL(V) \) orbits on symmetric bilinear forms on an \( n \)-dimensional \( V \) is in one-to-one correspondence with expressions \( n = p + q + r \), with each of \( p, q, \) and \( r \) a nonnegative integer. Consequently the total number of orbits is

\[
\frac{1}{r=n} + \frac{2}{r=n-1} + \cdots + \frac{n+1}{r=0} = (n+1)(n+2)/2.
\]

The symmetric matrix corresponding to the form in the theorem is

\[
\begin{pmatrix}
I_p & 0 & 0 \\
0 & -I_q & 0 \\
0 & 0 & 0_r
\end{pmatrix}
\]

the last indicated diagonal block is the \( r \times r \) zero matrix.

Sylvester's law is often expressed by saying that the number of positive and negative eigenvalues of a symmetric matrix is unchanged by the \( GL(n, \mathbb{R}) \) action described in the problem set. To me this seems misleading: the matrix of a symmetric form should be thought of not as a map from \( V \) to \( V \), but as a map from \( V \) to the dual space \( V^* \). From this point of view it doesn't make sense to talk about eigenvalues.

2. If \( \beta \) is a symmetric bilinear form on \( V \), the *orthogonal group of \( \beta \) is by definition*

\[
O_\beta = \{ g \in GL(V) \mid g \cdot \beta = \beta \}.
\]

(If the characteristic of \( k \) is two, this definition still makes sense but it is not correct—that is, it is not what is called an orthogonal group in characteristic two.)

If \( V = \mathbb{R}^n \), and \( \beta \) corresponds to a symmetric matrix \( B \), then the definition says that

\[
O_\beta = \{ g \in GL(n, \mathbb{R}) \mid t^g^{-1}Bg^{-1} = B \}.
\]
Find a simple description for the set of matrices $X$ in the Lie algebra $\mathfrak{o}_\beta$ of $O_\beta$. (Hint: you can use the fact that the Lie algebra of a Lie subgroup $H$ of $G$ consists of all $X \in \mathfrak{g}$ such that $\exp(tX) \in H$ for every real $t$.)

According to the hint, we want to decide when

$$t[\exp(-sX)]B \exp(-sX) = B.$$ 

for all real $s$. The power series formula for $\exp$ shows the transpose of the exponential is the exponential of the transpose; so the equation becomes

$$[\exp(-s^tX)]B = B \exp(sX).$$

Differentiating with respect to $s$ and setting $s = 0$, we see that this equation can hold only if

$$(*) -tXB = BX.$$ 

Since $B$ is assumed symmetric, $(*$) is equivalent to

$$t[BX] = BX,$$

or in turn

$$(**) BX \text{ is skew-symmetric.}$$

Conversely, from $(*)$ we can deduce by induction

$$(-tX)^kB = BX^k;$$

and summing over $k$ (with $s^k/k!$) gives

$$[\exp(-s^tX)]B = B \exp(sX).$$

This proves that

$$\mathfrak{o}_\beta = \{X \in \mathfrak{gl}(n, \mathbb{R}) \mid BX \text{ is skew-symmetric}\}.$$ 

3. Let $k = \mathbb{R}$, $n = p + q$, and let $B = \begin{pmatrix} I_p & 0 \\ 0 & -I_q \end{pmatrix}$ correspond to the symmetric form $\beta$ of signature $(p, q)$. The corresponding orthogonal group $O_\beta$ (preserving the “length” function

$$x_1^2 + \cdots + x_p^2 - x_{p+1}^2 - \cdots - x_{p+q}^2$$

is called $O(p, q)$. How many connected components does $O(p, q)$ have? (Hint: this is not so easy. You should look carefully at some small values of $p$ and $q$ to try to formulate the answer.)

The answer is one if $p = q = 0$ (this is obvious; the group has just one element, the empty matrix), two if exactly one of $p$ and $q$ is zero, and four if $p$ and $q$ are both nonzero. Here are some explicit calculations of the groups, to try to make the connected components clear:

$$O(1) = \{ (\pm 1) \}, \quad O(2) = \left\{ \begin{pmatrix} \cos \theta & \mp \sin \theta \\ \sin \theta & \pm \cos \theta \end{pmatrix} \mid \theta \in \mathbb{R} \right\}.$$
An element of $O(2)$ has as columns an orthonormal basis of $\mathbb{R}^2$. The first column $\begin{pmatrix} \cos \theta \\ \sin \theta \end{pmatrix}$ is what an arbitrary unit vector in $\mathbb{R}^2$ looks like. Orthogonal to that unit vector is the line through $\begin{pmatrix} -\sin \theta \\ \cos \theta \end{pmatrix}$; the possible second columns are the two unit vectors on that orthogonal line. For both $O(1)$ and $O(2)$ it is clear that there are exactly two connected components, consisting of the elements of determinant +1 and those of determinant −1.

Parallel reasoning for the quadratic form $x^2 - y^2$ gives

$$O(1,1) = \left\{ \begin{pmatrix} \epsilon \cosh t & \delta \sinh t \\ \epsilon \sinh t & \delta \cosh t \end{pmatrix} \mid t \in \mathbb{R}, \ \epsilon, \delta = \pm 1 \right\}.$$  

We need the possible extra sign in the first column because the hyperbolic cosine and sine give only the unit vectors whose first coordinate is positive. There are four connected components, corresponding to the four choices for the signs.

We have therefore shown that

\[(\text{base})\quad \pi_0(O(1)) = \pi_0(O(2)) = \mathbb{Z}/2\mathbb{Z}, \quad \pi_0(O(1,1)) = (\mathbb{Z}/2\mathbb{Z})^2.\]

I am going to write a (somewhat long but very explicit) decomposition of $O(p,q)$ into four (or two, or one) open and closed pieces; but I don’t see an easy way to prove that each of the pieces is connected. At the end I will write a (rather short inductive (in $p$ and $q$) proof that the component group $\pi_0(O(p,q))$ has order at most four (or two, or one). Putting those two pieces together will solve the problem.

Write

$$\mathbb{R}^n = \mathbb{R}^p \oplus \mathbb{R}^q,$$

an orthogonal decomposition with respect to $\beta$ into a positive definite and a negative definite subspace. The projection

$$\pi: \mathbb{R}^n \to \mathbb{R}^p, \quad \ker \pi = \mathbb{R}^q$$

has a negative definite kernel.

Write a typical element of $O(p,q)$ as

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

with $a$ a $p \times p$ matrix, $b$ a $p \times q$ matrix, and so on. By definition of $O(p,q)$, the $p + q$ columns of $g$ must be orthogonal vectors, the first $p$ having length 1 (in the quadratic form $\beta$) and the last $q$ having length $-1$. Write

$$P(g) = \text{span} \text{ of first } p \text{ columns of } g;$$

according to the previous statements the columns of $g$ are an orthonormal basis of $P(g)$, so $\beta$ restricted to $P(g)$ must be positive definite. The projection

$$\pi|_{P(g)}: P(g) \to \mathbb{R}^p$$

has kernel $P(g) \cap \mathbb{R}^q$, where $\beta$ is negative definite; so this kernel must be zero, and $\pi|_{P(g)}$ must be injective. Since $P(g)$ has dimension $p$,

$$\pi|_{P(g)} \text{ must be an isomorphism.}$$

(This argument is just the proof of “any positive definite subspace has dimension at most $p$” in Sylvester’s law of inertia.)
The matrix of $\pi|_{P(g)}$ in the given basis for $P(g)$ and the standard basis for $\mathbb{R}^p$ is the upper left block $a$ of $g$. We conclude that

$$\det(a) \neq 0 \quad \text{(any} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in O(p,q))$$

and in exactly the same way that $\det(d) \neq 0$. We have now defined two continuous functions

$$\text{sgn}(\det(a)), \quad \text{sgn}(\det(d)): O(p,q) \to \{\pm 1\}.$$ 

Using elements of $O(p,q)$ like

$$\begin{pmatrix} -1 & 0 & 0 \\ 0 & I_{p-1} & 0 \\ 0 & 0 & I_q \end{pmatrix}$$

shows that this pair of functions takes all four possible values (if $p$ and $q$ are both nonzero). In this way we conclude that

$$|\pi_0(O(p,q))| \geq \begin{cases} 4 & (p,q \geq 1) \\ 2 & (p > 0 \text{ and } q = 0 \text{ or } p = 0 \text{ and } q > 0) \\ 1 & (p = q = 0). \end{cases}$$

To finish the proof in this way, we must show that if $a$ and $d$ have positive determinant, then $g$ is in the identity component of $O(p,q)$. That’s what I don’t know an easy direct way to do.

Here, using results from class, is a derivation of upper bounds on component groups. The group $O(p,q)$ acts on $\mathbb{R}^{p,q}$. A theorem of Witt (for arbitrary orthogonal groups over arbitrary fields) says that the action on vectors of a fixed nonzero length is transitive. If $p > 0$, it follows that

vectors of length 1 = $O(p,q) \cdot e_1 \simeq O(p,q)/\{\text{stabilizer of } e_1\}$.

The vectors of length 1 are the $p+q-1$-dimensional hyperboloid

$$X_{p,q}^+ = \{(x_1, \ldots, x_{p+q} \mid x_1^2 + \cdots + x_p^2 - x_{p+1}^2 - \cdots - x_{p+q}^2 = 1\}.$$

The stabilizer of $e_1$ is more or less obviously equal to the orthogonal group of

$$\langle e_1 \rangle^\perp \simeq \mathbb{R}^{p-1,q},$$

so we get

$$X_{p,q}^+ \simeq O(p,q)/O(p-1,q).$$

I stated in class that

$$(*) \quad \pi_0(X_{p,q}^+) = \pi_0(O(p,q))/\{\text{image of } \pi_0(O(p-1,q))\}.$$

So what are the components of the hyperboloid? A point on the hyperboloid is given precisely by

1. a real number $r \geq 1$;
2. a vector $(x_1, \ldots, x_p) \in \mathbb{R}^p$ of length $r$; and
3. a vector $(x_{p+1}, \ldots, x_{p+q})$ of length $\sqrt{r^2-1}$. 

By multiplying the second vector by \((1 - t) \in [0, 1]\) and the first by \(\sqrt{r^2(1-t)^2 - (t^2 - 2t)}\), we get a (continuous) path connecting our point on the hyperboloid to a point on the unit sphere \(S^{p-1}\) in \(\mathbb{R}^p\). In this way we see that for any \(p \geq 1\)

\[
\pi_0(X^+_{p,q}) = \pi_0(S^{p-1}) = \begin{cases} 
\text{point} & (p \geq 2) \\
\text{two points} & (p = 1).
\end{cases}
\]

(In fact this proves that \(S^{p-1}\) is what’s called a deformation retract of \(X^+_{p,q}\); so the two spaces have the same homotopy type, and in particular all the homotopy groups are the same.)

Plugging this into (*) gives

\[
\pi_0(O(p, q)) = \begin{cases} 
\text{image of } \pi_0(O(p - 1, q)) & (p \geq 2) \\
\text{two cosets of image of } O(q) & (p = 1).
\end{cases}
\]

Combining (**) with (base) gives

\[
\left| \pi_0(O(p, q)) \right| \leq \begin{cases} 
4 & (p, q \geq 1) \\
2 & (p > 0 \text{ and } q = 0 \text{ or } p = 0 \text{ and } q > 0) \\
1 & (p = q = 0).
\end{cases}
\]

Together (upper) and (lower) give the result stated at the beginning.

4. Describe the Lie algebra \(\mathfrak{o}(p, q)\) by saying exactly which matrices

\[
\begin{pmatrix} A & B \\ C & D \end{pmatrix} \quad (A p \times p, \quad B p \times q, \quad C q \times p, \quad D q \times q)
\]

belong to the Lie algebra. (This means writing some conditions on and relations among the four matrices \(A, B, C,\) and \(D\). A good answer is one that can be understood just by knowing about matrices, without knowing about bilinear forms.) Use your description to calculate the dimension of \(O(p, q)\).

According to Problem 3, the symmetric bilinear form for \(O(p, q)\) corresponds to the matrix

\[
E_{p,q} = \begin{pmatrix} I_p & 0 \\ 0 & -I_q \end{pmatrix}.
\]

According to the solution to Problem 2, the Lie algebra therefore consists of matrices \(X\) such that \(E_{p,q}X\) is skew-symmetric. We calculate

\[
E_{p,q}X = \begin{pmatrix} I_p & 0 \\ 0 & -I_q \end{pmatrix} \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} A & C \\ -B & -D \end{pmatrix}.
\]

This last matrix is skew-symmetric if and only if \(A\) and \(D\) are skew-symmetric, and \(C = ^tB\).

A basis for the Lie algebra is therefore

\[
\begin{align*}
e_{ij} - e_{ji} & \quad 1 \leq i < j \leq p \\
e_{ir} + e_{ri} & \quad 1 \leq i \leq p, \quad p + 1 \leq r \leq p + q \\
e_{rs} - e_{sr} & \quad p + 1 \leq r < s \leq p + q
\end{align*}
\]

The total number of basis vectors is

\[
p(p - 1)/2 + pq + q(q - 1)/2 = [p^2 - p + 2pq + q^2 - q]/2 = [(p + q)^2 - (p + q)]/2.
\]

Since \(n = p + q\), the dimension is \(n(n - 1)/2\) (exactly the same as the dimension for the orthogonal Lie algebra \(\mathfrak{o}(n)\)).