18.755 fifth problem set solutions

1. The Lie group \( G = GL(n, \mathbb{R}) \) (of all invertible \( n \times n \) real matrices) has Lie algebra \( g = gl(n, \mathbb{R}) \) (all \( n \times n \) matrices, with Lie bracket given by commutator). The exponential map is

\[
\exp(X) = \sum_{j=0}^{\infty} \frac{X^j}{j!},
\]

the usual matrix exponential. (All those things you can assume.)

Given an (invertible) matrix \( T \in GL(n, \mathbb{R}) \), how do you tell whether there is a matrix \( X \) such that \( T = \exp(X) \)?

The theory of Jordan canonical form says that if \( A \in \text{Hom}(V, V) \) is a linear transformation of a finite-dimensional real vector space \( V \), then \( V \) may be written as a direct sum of subspaces preserved by \( A \), on which the restriction of \( A \) takes the following forms:

1. Subspace is \( \mathbb{R}^m \), and restriction of \( A \) is

\[
J_{\mathbb{R}}(\lambda, m) = \begin{pmatrix}
\lambda & 1 & 0 & \cdots & 0 \\
0 & \lambda & 1 & \cdots & 0 \\
& & \ddots & & \\
0 & \cdots & \lambda & 1 \\
0 & \cdots & 0 & \lambda
\end{pmatrix},
\]

Here \( \lambda \) can be any real number. The (generalized) eigenvalues of \( J_{\mathbb{R}}(\lambda, m) \) are all equal to \( \lambda \), with (generalized) multiplicity \( m \).

2. Subspace is \( \mathbb{C}^m \) regarded as a \( 2m \)-dimensional real vector space, and restriction of \( A \) is

\[
J_{\mathbb{C}}(\lambda, m) = \begin{pmatrix}
\lambda & 1 & 0 & \cdots & 0 \\
0 & \lambda & 1 & \cdots & 0 \\
& & \ddots & & \\
0 & \cdots & \lambda & 1 \\
0 & \cdots & 0 & \lambda
\end{pmatrix},
\]

a complex-linear transformation regarded as real-linear. Here \( \lambda \) can be any non-real complex number. The generalized eigenvalues of \( J_{\mathbb{C}}(\lambda, m) \) as a real linear transformation are \( \lambda \) and \( \overline{\lambda} \), each with multiplicity \( m \).

The number and types of the summands in this decomposition of \( V \) are uniquely determined by \( A \) (up to permutation). Of course the second type makes sense also if \( \lambda \) is real; but

\[
J_{\mathbb{C}}(\lambda, m) \simeq J_{\mathbb{R}}(\lambda, m) \oplus J_{\mathbb{R}}(\lambda, m) \quad (\lambda \in \mathbb{R}).
\]

This formulation of Jordan canonical form is not perhaps so close to what you might find in a linear algebra book; but it’s equivalent, and it’s not so difficult to make a version for any perfect ground field \( k \). (I forget what might be required to deal with matrices whose characteristic polynomials correspond to inseparable extensions.)
In order to calculate the exponential of an arbitrary matrix $X$, we write it as a sum of Jordan blocks and calculate the exponential of each block separately. This is easy:

$$\exp(J_{R}(\lambda, m)) = e^{\lambda} \begin{pmatrix}
1 & 1 & 1/2! & 1/3! & \cdots & 1/m! \\
0 & 1 & 1/2! & 1/(n-1)! & \cdots & 1/n! \\
\vdots & & & & & \\
0 & 0 & 0 & 0 & \cdots & 1
\end{pmatrix}.$$ 

It’s a pretty easy exercise to see that this matrix is conjugate to $J_{R}(e^{\lambda}, m)$. That is, each real Jordan block for $X$ gives a real Jordan block for $\exp(X)$ with a strictly positive real eigenvalue.

In exactly the same way, we calculate

$$\exp(J_{C}(\lambda, m)) \simeq J_{C}(e^{\lambda}, m).$$

Every nonzero nonreal complex number is the exponential of some nonreal complex number; so every complex Jordan block for a real invertible matrix is the exponential of a complex Jordan block of the same size.

The tricky part is that $e^{\lambda}$ can be real even if $\lambda$ is not. We find

$$\exp((J_{C}(a + 2q\pi i, m)) \simeq J_{R}(e^{a}, m) \oplus J_{R}(e^{a}, m) \quad (q \in \mathbb{Z}).$$

$$\exp((J_{C}(a + (2q + 1)\pi i, m)) \simeq J_{R}(-e^{a}, m) \oplus J_{R}(-e^{a}, m) \quad (q \in \mathbb{Z}).$$

In the first case, we are just getting a different way to write a matrix that we already knew was an exponential. For $GL(2, \mathbb{R})$, we now can write in two ways

$$I = \exp \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = \exp \begin{pmatrix} 0 & 2\pi \\ -2\pi & 0 \end{pmatrix}. $$

But in the second case, we are seeing how to write as exponentials some matrices with negative eigenvalues, which previously we did not know how to do. For example, for $GL(2, \mathbb{R})$,

$$-I = \exp(J_{C}(i\pi, 1)) = \exp \begin{pmatrix} 0 & \pi \\ -\pi & 0 \end{pmatrix}. $$

So here is the answer to the question in the problem: given an invertible matrix $T \in GL(n, \mathbb{R})$, change bases to write it as a sum of Jordan blocks as described above. Then $T$ can be written as an exponential if and only if for every negative real eigenvalue of $T$, each Jordan block size occurs an even number of times. (For matrices that are diagonalizable over $\mathbb{C}$, the condition is just that every negative real eigenvalue must have even multiplicity.) We’ve even said fairly explicitly how to find a logarithm of $T$, from knowledge of the Jordan canonical form.

2. If $G$ is any topological group, the identity component $G_{e}$ is by definition the largest connected subgroup. (There is a pretty easy argument that such a subgroup exists and (I think!) that it is closed.) If $G$ is a Lie group,
then $G_e$ is the connected subgroup attached to the Lie (sub)algebra $\mathfrak{g}$ of $G$. (All those things you can assume.)

**Describe explicitly the identity component of $GL(n, \mathbb{R})$.**

This is the subgroup generated by the image of the exponential map. I claim that the subgroup generated by the image of exp consists of all matrices with strictly positive determinant. Here is why. First we have to prove that everything in the image of exp has strictly positive determinant. It’s enough to do this one Jordan block at a time, since the determinant of a block diagonal matrix is the product of the determinants of the blocks. The two calculations are

$$\det J\mathbb{R}(e^\lambda, m) = e^{m\lambda} > 0 \quad (\lambda \in \mathbb{R})$$

and

$$\det J\mathbb{C}(e^\lambda, m) = |e^{m\lambda}|^2 > 0 \quad (\lambda \in \mathbb{C}).$$

For the second, I talked a long time ago in class about the fact that if $A$ is an $m \times m$ complex matrix regarded as a $2m \times 2m$ real matrix, then $\det_{\mathbb{R}}(A) = |\det_{\mathbb{C}}(A)|^2$.

Conversely, we want to show that if $T$ is any matrix of positive determinant, then it’s in the subgroup generated by the image of exp. Suppose first that $S^2 = I$ is a matrix having only the eigenvalues $+1$ and $-1$; say

$$\mathbb{R} = V_+ \oplus V_-$$

is the eigenspace decomposition of $S$. Then

$$\det S = (-1)^{\dim V_+}.$$

If the $-1$ eigenspace is even dimensional, then the discussion in Problem 1 shows how to write $S$ as an exponential; so in that case it belongs to the identity component.

Now suppose $T$ is arbitrary invertible. Write

$$\mathbb{R}^n = W_+ \oplus W_- \oplus W_C,$$

with $W_+$ the sum of the Jordan blocks for positive eigenvalues; $W_-$ the sum of the Jordan blocks for negative eigenvalues; and $W_C$ the sum of the Jordan blocks for nonreal eigenvalues. Then

$$\text{sgn}(\det(T)) = (-1)^{\dim W_-}.$$

Assume now that $T$ has positive determinant, so that $\dim W_-$ is even; we want to show that $T$ is in the subgroup generated by the image of exp. Define $S$ to have $-1$ eigenspace $W_-$ and $+1$ eigenspace $W_+ \oplus W_C$. Then $S$ is in the image of exp, so it is enough to show that $ST$ is in the image of exp. But the construction of $S$ shows that the eigenvalues of $ST$ are precisely the eigenvalues of $T$, with the negative ones replaced by their absolute values. Therefore every real eigenvalue of $ST$ is strictly positive, so (Problem 1 again) $ST$ is an exponential (even one Jordan block at a time).

We showed that anything in the identity component is a product of at most two exponentials. This turns out to be true for any semisimple Lie group $G$. I’m not certain whether it’s true for arbitrary Lie groups.
3. (In this problem the field \( k \) can be arbitrary.) You know that a basis of an \( n \)-dimensional vector space \( V \) is a linearly independent spanning sequence \( (v_1, \ldots, v_n) \). Define

\[ B(V) = \text{set of all bases of } V. \]

The group \( GL(V) \) of invertible linear transformations of \( V \) acts on \( B(V) \) by

\[ g \cdot (v_1, \ldots, v_n) = (gv_1, \ldots, gv_n). \]

A flag in \( V \) is a chain of subspaces

\[ \{0\} = V_0 \subset V_1 \subset \cdots \subset V_n = V, \quad \dim V_j = j. \]

Define

\[ F(V) = \text{set of all flags in } V. \]

Prove that the action of \( GL(V) \) on \( B(V) \) is simply transitive: that if \( b_1 \) and \( b_2 \) are any two bases of \( V \), then there is a unique \( g \in V \) such that \( gb_1 = b_2 \).

Prove that the action of \( GL(V) \) on \( F(V) \) is transitive.

Since all \( n \)-dimensional vector spaces are isomorphic, we might as well assume \( V = k^n \), thought of as column vectors. More precisely, we do this by sending the basis \( b_1 \) of \( V \) to the standard basis \((e_1, e_2, \ldots, e_n)\) of \( k^n \).

An \( n \times n \) matrix is a list of \( n \) vectors; the matrix \( g \) is invertible if and only if the list of vectors is a basis of \( k^n \). That is, we can identify \( B(k^n) \) with \( GL(n, k) \). In this identification, the basis \( b_2 \) corresponding to \( g \) is precisely \( g \cdot b_1 \), because the columns of \( g \) arise by applying \( g \) to the standard basis vectors. This proves the simple transitivity.

There is a natural map

\[ B(V) \to F(V), \quad (v_1, \ldots, v_n) \mapsto (\langle \rangle \subset \langle v_1 \rangle \subset \langle v_1, v_2 \rangle \subset \cdots). \]

This map respects the action of \( GL(V) \) on each set; that’s what “natural” means. The map is surjective, because given a flag you can construct a basis step by step: start with the empty basis of \( V_0 \), extend to a basis \((v_1)\) of \( V_1 \), then extend to a basis \((v_1, v_2)\) of \( V_2 \), and so on. (This uses the fact that a basis of a subspace can always be extended to a basis of the whole space.)

The surjectivity of the map and the transitivity of the \( GL(V) \) action on \( B(V) \) imply the transitivity of the action on \( F(V) \).