18.755 fifth problems, due Monday, October 19, 2015

1. The Lie group \( G = GL(n, \mathbb{R}) \) (of all invertible \( n \times n \) real matrices) has Lie algebra \( g = \mathfrak{gl}(n, \mathbb{R}) \) (all \( n \times n \) matrices, with Lie bracket given by commutator). The exponential map is

\[
\exp(X) = \sum_{j=0}^{\infty} \frac{X^j}{j!},
\]

the usual matrix exponential. (All those things you can assume.)

Given an (invertible) matrix \( T \in GL(n, \mathbb{R}) \), how do you tell whether there is a matrix \( X \) such that \( T = \exp(X) \)? (An answer might be something like, “such a matrix \( T \) exists if and only if the eigenvalues of \( T \) are between 0 and 2. The proof is that the power series

\[
X = \log(T) = (T - I) - (T - I)^2/2 + (T - I)^3/3 - \ldots
\]

has radius of convergence one.”)

2. If \( G \) is any topological group, the identity component \( G_e \) is by definition the largest connected subgroup. (There is a pretty easy argument that such a subgroup exists and (I think!) that it is closed.) If \( G \) is a Lie group, then \( G_e \) is the connected subgroup attached to the Lie (sub)algebra \( g \) of \( g \). (All those things you can assume.)

Describe explicitly the identity component of \( GL(n, \mathbb{R}) \).

3. (In this problem the field \( k \) can be arbitrary.) You know that a basis of an \( n \)-dimensional vector space \( V \) is a linearly independent spanning sequence \((v_1, \ldots, v_n)\). Define

\[
B(V) = \text{set of all bases of } V.
\]

The group \( GL(V) \) of invertible linear transformations of \( V \) acts on \( B(V) \) by

\[
g \cdot (v_1, \ldots, v_n) = (gv_1, \ldots, gv_n).
\]

A flag in \( V \) is a chain of subspaces

\[
\{0\} = V_0 \subset V_1 \subset \cdots \subset V_n = V, \quad \dim V_j = j.
\]

Define

\[
F(V) = \text{set of all flags in } V.
\]

Prove that the action of \( GL(V) \) on \( B(V) \) is simply transitive: that if \( b_1 \) and \( b_2 \) are any two bases of \( V \), then there is a unique \( g \in V \) such that \( gb_1 = b_2 \).

Prove that the action of \( GL(V) \) on \( F(V) \) is transitive.