18.755 Problem Set 3 solutions

The first two problems refer to the notes GLUE “Gluing manifolds together” on the course web page (near the bottom).

1. Suppose $N$ is constructed from two manifolds $M_1 \supset U_1$ and $M_2 \supset U_2$ as in GLUE, (2.1). Find an example in which $N$ is not a manifold. (The definition of manifold is the one given in class, and appearing in the books of Conlon, Munkres, Warner, ... , and on the wikipedia page “Differentiable manifold.”)

The notes state that $N$ is locally diffeomorphic to open sets in Euclidean space, and the fact that $N$ is separable is an immediate consequence of the separability of $M_1$ and $M_2$. The only remaining axiom is Hausdorff; so we must find an example in which $N$ has points $n_1$ and $n_2$ that do not have disjoint neighborhoods. If both $n_i$ lie in $M_1$, then they have disjoint neighborhoods in $M_1$ since $M_1$ is Hausdorff; and these will be neighborhoods in $N$ by definition of the topology. Similarly if both lie in $M_2$.

Perhaps after relabelling, we may assume

$$n_1 \in M_1 \setminus M_2 = M_1 - U_1, \quad n_2 \in M_2 \setminus M_1 = M_2 - U_2.$$  

If $n_1$ has a neighborhood $V_1 \subset M_1$ not meeting $U_1$, then $V_1$ and any neighborhood $V_2$ of $n_2$ in $M_2$ must be disjoint. Similarly if $n_2$ has a neighborhood not meeting $U_2$. The only possibly difficulty therefore arises if

$$n_1 \in \overline{U_1} - U_1 = \partial U_1, \quad n_2 \in \partial U_2.$$  

The Hausdorff condition will fail if and only if for every neighborhood $V_i$ of $n_i$ in $M_i$, 

$$\phi_{12}(V_1 \cap U_1) \text{ meets } V_2 \cap U_2.$$  

What this means is that we need a sequence in $n_1^j \in U_1$ converging to the boundary point $n_1$, such that $\phi_{12}(n_1^j)$ converges in $U_2$ to the boundary point $n_2$.

An easy way to achieve this is with $M_1 = M_2 = \mathbb{R}$, $U_1 = U_2 = \mathbb{R}^x$ (the nonzero reals), $\phi_{12}$ the identity map, and $n_1 = n_2 = 0$. The topological space $N$ looks like $\mathbb{R}$ except that it has two origins $0_1$ and $0_2$. A neighborhood base at $0_1$ is

$$(-\epsilon, 0) \cup \{0_1\} \cup (0, \epsilon) \quad (\epsilon > 0)$$

and a neighborhood base at $0_2$ is

$$(-\delta, 0) \cup \{0_2\} \cup (0, \delta) \quad (\delta > 0).$$

Of course any two of these neighborhoods overlap, so $N$ is not Hausdorff.

As long as we are looking at nasty examples, a smooth vector field on $N$ is given by a smooth vector field on $\mathbb{R}$; its values at $0_1$ and $0_2$ are necessarily “the same.” Integral curves of the vector field $d/dx$ satisfying an initial condition $\gamma(t_0) = x_0$ are not unique unless $x_0 = 0$; the value at $t = t_0 - x_0$ may be chosen to be either $0_1$ or $0_2$. (You can find a uniqueness theorem for integral curves claiming to cover this example in a moderately famous differential geometry text. Can’t trust everything you read.)

2. Give an example in which the construction of (4.1) in GLUE does not give a manifold.

According to the analysis in the solution to Problem 1, we are looking for $n_1$ so that the integral curve $\gamma_{n_1}^1$ in $M_1$ is not defined at $-\epsilon$, but $\gamma_{n_1}^1$ is defined at $-\epsilon$ for a sequence $n_1^j$ converging to $n_1$. Then we want to define $n_2^j = \gamma_{n_1}^1(-\epsilon)$, and ask that the sequence $n_2^j$ converge to $n_2$ so that $\gamma_{n_2}^2$ is not defined at $-\epsilon$.

The easiest way to arrange the first requirement is for the maximal integral curve $\gamma_{n_1}$ to be defined on $(-\epsilon, b_1)$. Then we can choose $n_1^j = \gamma_{n_1}(1/j)$ (which makes sense at least for large $j$). The integral curve through $n_1^j$ is defined on $(-\epsilon - 1/j, b_1 - 1/j)$, so these points indeed belong to $U_1$. Applying $\phi_{12}$ gives

$$n_2^j = \gamma_{n_1}(-\epsilon + 1/j).$$
If this sequence has a limit point $n_2$, then it’s not hard to see that the integral curve must actually be defined at $-\epsilon$, satisfying 
$$\gamma_{n_1}(-\epsilon) = n_2.$$ 
This contradicts our original choice of $n_1$.

So we need to find a more subtle way to arrange matters: roughly, that $\gamma_{n_1}$ should not be defined close to $-\epsilon$, even though nearby integral curves are defined. One possibility is to choose 
$$M = \mathbb{R}^2 \setminus \{0\}, \quad X = \frac{\partial}{\partial x}.$$ 

Then the maximal integral curves are 
$$\gamma_{x_0,y_0}(t) = (x_0 + t, y_0) \begin{cases} 
  t \in (-\infty, \infty), & (y_0 \neq 0) \\
  t \in (-x_0, \infty), & (y_0 = 0, x_0 > 0) \\
  t \in (-\infty, -x_0), & (y_0 = 0, x_0 < 0). 
\end{cases}$$

Then 
$$U_1 = U_{\geq -\epsilon} = \mathbb{R}^2 - [0, \epsilon] \times \{0\}, \quad U_2 = V_{\leq \epsilon} = \mathbb{R}^2 - [-\epsilon, 0] \times \{0\}.$$ 

We have 
$$\partial U_1 = (0, \epsilon] \times \{0\}, \quad \partial U_2 = (-\epsilon, 0) \times \{0\}.$$ 

All of the pairs 
$$n_1, n_2 = n_1 - (\epsilon, 0)$$

lack disjoint neighborhoods. The space $N$ consists of $\mathbb{R}^2$ with two copies of the interval $(0, \epsilon) \times \{0\}$.

3. Suppose that $V$ is a finite-dimensional real vector space, and that 
$$\alpha: \mathbb{R} \times V \to V$$
is a continuous (not necessarily smooth) action of $\mathbb{R}$ on $V$ by linear transformations. It is equivalent to assume that 
$$A: \mathbb{R} \to GL(V), \quad A(t)v = \alpha(t, v)$$
is a continuous group homomorphism. Prove that there is a linear map $T \in \text{Hom}(V, V)$ with the property that 
$$A(t) = \exp(tT).$$

According to the hint, we would like to prove that $t \mapsto A(t)$ is a smooth map. Suppose we know that. Because $GL(V)$ is an open subset of the vector space $\text{Hom}(V, V)$, we have 
$$T_g GL(V) = \text{Hom}(V, V) \quad (g \in GL(V)).$$

The differential of $A(t)$ at $t = 0$ (which exists by the assumed smoothness of $A$) is 
$$T = \lim_{t \to 0} \frac{A(t) - A(0)}{t - 0} \in \text{Hom}(V, V).$$

The differential at $t = s$ is 
$$\lim_{t \to 0} \frac{A(t + s) - A(s)}{t - 0} = \left( \lim_{t \to 0} \frac{A(t) - A(0)}{t - 0} \right) A(s) = TA(s) \in \text{Hom}(V, V).$$
Therefore $A(s)$ satisfies the differential equation
\[
\frac{dA}{dt}(t) = TA(t), \quad A(0) = I;
\]
the solution (almost by definition of the exponential) is
\[
A(t) = \exp(tT).
\]

So it is indeed enough to prove that $A(t)$ is smooth. A map into $\mathbb{R}^N$ is smooth if and only if all the $N$ (real-valued) coordinates of the map are smooth. Therefore a map into matrices is smooth if and only if each matrix entry is smooth, and this is true if and only if each column is a smooth function. The columns of $A(t)$ (once we choose a basis of $V$) are the functions $A(t)e_i$, and the other functions $A(t)v$ are linear combinations of the column functions. This proves that $A(t)$ is smooth if and only if each $A(t)v$ is smooth.

Let us call a vector $w \in V$ smooth if the function $A(t)w$ is smooth. (Of course this property of $w$ depends enormously on the function $A(t)$, and the terminology obscures this; but it’s the standard terminology.) Since the maps $A(t)$ are each linear, it follows immediately that the collection of smooth vectors is a subspace of $V$.

The hint asked you to prove that for any $v \in V$ and $\phi \in C^\infty_c(\mathbb{R})$, the vector
\[
\omega(v, \phi) = \int_{-\infty}^{\infty} \phi(t)A(t)v \, dt
\]
is smooth. This can be done more or less along the lines of one of the problems last week. We have
\[
A(s)\omega(v, \phi) = \int_{-\infty}^{\infty} \phi(t)A(s)A(t)v \, dt
= \int_{-\infty}^{\infty} \phi(t)A(s+t)v \, dt
= \int_{-\infty}^{\infty} \phi(t'-s)A(t')v \, dt'
\]
(The first equality (taking the linear map inside the integral) is easy; the second is the fact that $A$ is a homomorphism; and the third is change of variable.) Now the last formula writes $A(s)\omega$ as an integral with a parameter $s$; and you can differentiate this formula in $s$ under the integral sign (just like last week). The conclusion is that $A(s)w$ is a smooth function of $s$, and therefore that $w$ belongs to the subspace of smooth vectors in $V$.

We are asked to show that every vector in $V$ is smooth. It’s enough to show that every vector in $V$ is of the form $\omega(v, \phi)$ for some $v \in V$ and $\phi \in C^\infty_c(\mathbb{R})$. If $w = 0$, then $w = \omega(0, \phi)$, and we are done; so assume $w \neq 0$. The equation
\[
w = \int_{-\infty}^{\infty} \phi(t)A(t)v \, dt
\]
is a painful equation to solve for $v$ and $\phi$. So we look just for an approximate solution. Fix a norm (like Euclidean length in some basis $\| \cdot \|$ on $V$) giving a corresponding operator norm on $\text{Hom}(V, V)$ satisfying
\[
\|Sv\| \leq \|S\| \cdot \|v\|.
\]
Given $w$ and any $\epsilon > 0$, we want to find $v$ so that
\[
\left\| \int_{-\infty}^{\infty} \phi(t)A(t)v \, dt - w \right\| \leq \epsilon.
\]
We will actually achieve this using \( v = w \); all the magic is in choosing \( \phi \). If \( w = 0 \), then \( \omega(0, \phi) = w \) for any \( \phi \); so we assume henceforth that \( w \neq 0 \). First choose \( \delta \) so small that

\[
\|A(t) - I\| = \|A(t) - A(0)\| < \epsilon / \|w\| \quad (|t| < \delta).
\]

(Recall that we are assuming \( w \neq 0 \).) Such a \( \delta \) exists because \( A \) is assumed to be continuous. Now

\[
\|A(t)w - w\| < \left( \frac{\epsilon}{\|w\|} \right) \|w\| = \epsilon.
\]

Next, choose \( \phi \in C^\infty_c(\mathbb{R}) \) so that

1. \( \phi \geq 0 \),
2. \( \phi \) is supported on \([-\delta, \delta]\), and
3. \( \int_{-\infty}^{\infty} \phi(t) \ dt = 1 \).

Then

\[
w = \int_{-\infty}^{\infty} \phi(t)A(0)w \ dt,
\]

so

\[
\left\| \int_{-\infty}^{\infty} \phi(t)A(t)w \ dt - w \right\| = \left\| \int_{-\infty}^{\infty} \phi(t)(A(t) - A(0))w \ dt \right\|
\]

\[
= \left\| \int_{-\delta}^{\delta} \phi(t)(A(t) - A(0))w \ dt \right\|
\]

\[
\leq \int_{-\delta}^{\delta} \phi(t) \| (A(t) - A(0))w \| \ dt
\]

\[
< \int_{-\delta}^{\delta} \phi(t) \epsilon \ dt = \epsilon.
\]

That is, we have shown that \( w \) may be approximated within any \( \epsilon > 0 \) by a smooth vector.

The smooth vectors are a linear subspace of \( V \), and no proper subspace of \( V \) can be dense. (This argument mostly works for infinite-dimensional \( V \); the most important difference in that case is that proper subspaces can be dense.) Consequently the smooth vectors must be all of \( V \), as we wished to show.

4. Suppose \( T \) is an \( n \times n \) real matrix. Find necessary and sufficient conditions on \( T \) for the one-parameter group

\[ \{ \exp(tT) \mid t \in \mathbb{R} \} \]

to be a closed subgroup of \( GL(n, \mathbb{R}) \). Here is the answer: \( \exp(\mathbb{R}T) \) is not closed if and only if \( T \) is diagonalizable as a complex matrix; and all the nonzero eigenvalues are purely imaginary numbers \( iy_j \); and some ratio \( y_j / y_k \) is irrational.

The proof requires some detailed understanding of Jordan canonical form for real matrices. I will just quote a useful version of this, without helping you find a reference for exactly this statement.

**Theorem.** Suppose \( T \) is a linear transformation on a finite dimensional real vector space \( V \). Then there is a unique decomposition

\[ T = T_h + T_e + T_n \]

subject to the requirements

1. the linear transformations \( T_h \), \( T_e \), and \( T_n \) commute with each other;
2. the linear transformation \( T_h \) is diagonalizable with real eigenvalues;
3. the linear transformation \( T_e \) is diagonalizable over \( \mathbb{C} \), with purely imaginary eigenvalues; and
4. the linear transformation \( T_n \) is nilpotent: \( T^n = 0 \) for some \( N > 0 \).
The subscripts \( h, e, \) and \( n \) stand for “hyperbolic,” “elliptic,” and “nilpotent.”

Suppose \( f \) is a continuous map from \( \mathbb{R} \) to a metric space. The image \( f(\mathbb{R}) \) can fail to be closed only if there is an \textit{unbounded} sequence of real numbers \( t_i \) such that \( f(t_i) \) converges in the metric space. (You should think carefully about why this is true: the proof is very short, but maybe not obvious.)

So if the image is not closed, then we can find an unbounded sequence \( t_i \) so that \( \exp(t_i T) \) is convergent in \( GL(n, \mathbb{R}) \), and in particular is a bounded sequence of matrices. By passing to a subsequence, we may assume that all \( t_i \) have the same sign. Since matrix inversion is a homeomorphism, \( \exp(-t_i T) \) is also a (convergent and) bounded sequence of matrices. Perhaps replacing the sequence by its negative, we may assume \( all \ t_i > 0 \).

Now the Jordan decomposition guarantees

\[
\exp(tT) = \exp(tT_h) \exp(tT_e) \exp(tT_n).
\]

In appropriate coordinates the matrix \( T_e \) is block diagonal with blocks

\[
\begin{pmatrix}
0 & y_j \\
-y_j & 0
\end{pmatrix}
\]

(with \( y_j \neq 0 \) and zeros; so \( \| \exp(tT_e) \| \) is bounded. The power series for \( \exp(tT_n) \) ends after the term \( t^N T^N / N! \); so \( \| \exp(tT_n) \| \) has polynomial growth in \( t \).

If \( T_h \) has a positive eigenvalue, then \( \exp(tT_h) \) grows exponentially in \( t \), so the sequence \( \exp(t_i T) \) cannot be bounded. Similarly, if \( T_h \) has a negative eigenvalue, then \( \exp(-t_i T) \) grows exponentially. The conclusion is that if the image is not closed, then \( T_h = 0 \).

In exactly the same way, suppose \( T_n^N \neq 0 \) but \( T_{n+1}^N = 0 \). Then \( \exp(tT_n) \) grows like a polynomial of degree \textit{exactly} \( N \); so (because of the boundedness of \( \exp(tT_e) \)) we conclude that \( \exp(tT) \) also grows like a polynomial of degree \textit{exactly} \( N \). The conclusion is that if the image is not closed, then \( N = 0 \), which means \( T_n = 0 \).

We have shown that the image can fail to be closed only if \( T = T_e \). In this case the image is bounded; so \textit{it is closed if and only if it is compact}. Suppose that the eigenvalues of \( T = T_e \) are \( iy_j \) as above, so that \( \exp(tT) \) has diagonal blocks

\[
\begin{pmatrix}
\cos(ty_j) & \sin(ty_j) \\
-\sin(ty_j) & \cos(ty_j)
\end{pmatrix}
\]

If all the ratios \( y_j / y_1 = p_j / q_j \) are rational, then it’s easy to see that \( \exp(tT) \) is periodic with period (dividing)

\[
(\text{least common multiple of all } q_j)(2\pi / y_1);
\]

so the image is a circle (or a point), and is closed.

Conversely, suppose that the image is compact. More or less the example done in class shows that the group

\[
\begin{pmatrix}
\cos(ty_1) & \sin(ty_1) \\
-\sin(ty_1) & \cos(ty_1)
\end{pmatrix}
\begin{pmatrix}
0 & \\
& 0
\end{pmatrix}
\begin{pmatrix}
\cos(ty_2) & \sin(ty_2) \\
-\sin(ty_2) & \cos(ty_2)
\end{pmatrix}
\]

is compact if and only if \( y_1 / y_2 \) is \textit{rational}. (I’m tired of typing, so I won’t write out a proof.) By projecting the (assumed compact) \{ \( \exp(tT) \) \} on various collections of four coordinates, and using “continuous image of compact is compact,” we deduce that all the ratios \( y_j / y_k \) are rational, as we wished to show.