18.700 Problem Set 9

Due in class Wednesday December 4 (changed from syllabus); late work will not be accepted. Your work on graded problem sets should be written entirely on your own, although you may consult others before writing.

1. (8 points) Suppose $V$ is a real or complex inner product space. A linear map $S \in \mathcal{L}(V)$ is called skew-adjoint if $S^* = -S$. Suppose $V$ is complex and finite-dimensional, and $S$ is skew-adjoint. Show that the eigenvalues of $S$ are all purely imaginary (that is, real multiples of $i$) and that there is an orthogonal direct sum decomposition

\[ V = \bigoplus_{\lambda \in \mathbb{R}} V_{i\lambda}. \]

2. (16 points) Suppose $V$ is an $n$-dimensional real inner product space, and $S$ is a skew-adjoint linear transformation of $V$.
   a) Show that $Sv$ is orthogonal to $v$ for every $v \in V$.
   b) Show that every eigenvalue of $S^2$ is a real number less than or equal to zero.
   c) Suppose (still assuming $S$ is skew-adjoint) that $S^2 = -I$ (the negative of the identity operator on $V$). Show that we can make $V$ into a complex inner product space, by defining scalar multiplication as 

\[ (a + bi)v = av + bSv \]

and the complex inner product as

\[ \langle v, w \rangle_C = \langle v, w \rangle - i\langle Sv, w \rangle. \]

What is the dimension of $V$ as a complex vector space?

 d) Now drop the assumption that $S^2 = -I$, but still assume $S$ is skew-adjoint. Show that there is an orthonormal basis of $V$ in which the matrix of $S$ is

\[
\begin{pmatrix}
  0 & -\lambda_1 \\
  \lambda_1 & 0 \\
  & \ddots \\
  & & 0 & -\lambda_r \\
  & & \lambda_r & 0 & \ddots \\
  & & & \ddots & 0
\end{pmatrix},
\]

with $\lambda_1 \geq \cdots \geq \lambda_r > 0$. That is, the matrix of $S$ in this basis is block diagonal, with $r$ $2 \times 2$ blocks of the form

\[
\begin{pmatrix}
  0 & -\lambda \\
  \lambda & 0 \\
\end{pmatrix}
\]

with $\lambda > 0$, and $n - 2r$ $1 \times 1$ blocks $(0)$. (Hint: first diagonalize $S^2$.)

3. (6 points) Give an example of a square complex matrix $A$ with the property that $A$ has exactly three distinct eigenvalues, but $A$ is not diagonalizable. (For full credit, you should prove that your matrix has the two required properties.)