

## 18.700 Problem Set 6

Due in class Tuesday November 4; late work will not be accepted. Your work on graded problem sets should be written entirely on your own, although you may consult others before writing.

1. (8 points) Suppose that we are given three polynomials

$$p_2(x) = ax^2 + bx + c, \quad p_1(x) = dx + e, \quad p_0(x) = f$$

with real coefficients. This problem is about the differential operator

$$D = p_2(x) \frac{d^2}{dx^2} + p_1(x) \frac{d}{dx} + p_0(x).$$

- Explain why the differential operator  $D$  preserves the  $m + 1$ -dimensional space  $\mathcal{P}_m(\mathbb{R})$  of real polynomials of degree less than or equal to  $m$ .
- Find all the diagonal entries of the matrix of  $D$  in the basis  $(1, x, x^2, \dots, x^m)$ . (Your answers will depend on the constants  $a, b, c, d, e, f$ .)
- Suppose that the constants satisfy  $0 < a < d$ . Prove that  $D$  is diagonalizable on  $\mathcal{P}_m(\mathbb{R})$ .
- Consider the special case

$$D = (x^2 - x) \frac{d^2}{dx^2} + (2x - 1) \frac{d}{dx}.$$

Find a basis of eigenvectors of  $D$  on the four-dimensional space  $\mathcal{P}_3(\mathbb{R})$ .

2. (8 points) This problem is about the inner product space  $C[0, 1]$  of real-valued continuous functions on the interval  $[0, 1]$ , with inner product

$$\langle p, q \rangle = \int_0^1 p(x)q(x)dx.$$

It's related to the examples 6.33 and 6.44 in the text about finding a good polynomial approximation to  $\cos(x)$ .

- Let  $U = \mathcal{P}_3(\mathbb{R})$  be the four-dimensional subspace of  $C[0, 1]$  consisting of polynomials of degree less than or equal to three. Apply the Gram-Schmidt process described in the notes on orthogonal bases on the course web site to convert the basis  $(1, x, x^2, x^3)$  of  $U$  to an orthogonal basis having the same span. (I suggest the notes because it's easier than the text.)
- Define  $f(x) = \sqrt{x}$ , regarded as a function in  $C[0, 1]$ . Calculate the orthogonal projection  $P_U(f)$  of  $f$  on the subspace  $U$ . (Your answer should be a cubic polynomial in  $x$  with rational numbers as coefficients.)
- Make a little table displaying the values of  $f(x)$  and  $p_U(f)(x)$  for at least the values  $x = 0, 1/4, 1/2, 3/4, 1$ , to three decimal places.
- Find some relationship between your answers to 1(d) and 2(a). Make a guess about how to generalize that relationship to higher degree polynomials.

3. (8 points) Axler, page 175, exercise 4.