18.02A PROBLEM SET 13 SOLUTIONS

1. (10 points) This problem concerns an analogue of the theory of conservative fields using surface integrals in place of line integrals.

a) Suppose that

$$\mathbf{F} = M(x,y,z)\mathbf{i} + N(x,y,z)\mathbf{j} + P(x,y,z)\mathbf{k}$$

is a vector field in space, and that each of the functions M, N and P has continuous second partial derivatives. Show that $\operatorname{div}\operatorname{curl}\mathbf{F} = 0$.

The vector field $\operatorname{curl}\mathbf{F}$ has three components, each of which is a difference of two partial derivatives of M, N, or P; for example, the first coordinate is $\partial P/\partial y - \partial N/\partial z$. The divergence is the sum of the partial derivative of the first coordinate with respect to x, the second with respect to y, and the third with respect to z. It is therefore a sum of six second partial derivatives of M, N, and P. The terms involving P are $\partial^2 P/\partial x\partial y$ and $-\partial^2 P/\partial y\partial x$. Since the second partial derivatives are continuous, it doesn't matter in which order you differentiate; so these two terms cancel. The same applies to the other four terms.

b) You might say (although no one does!) that a vector field G in space is *superficially conservative* if the flux of G through a surface S depends only on the boundary curve C of S. That is, if S and S' are two surfaces having the same boundary curve C, then

$$\iint_S \mathbf{G}\cdot d\mathbf{S} = \iint_{S'} \mathbf{G}\cdot d\mathbf{S}.$$

Show that if $\mathbf{G} = \operatorname{curl}\mathbf{F}$, then G is superficially conservative. (Hint: use Stokes' theorem.)

Write C for the common boundary curve of S and S'. Applying Stokes' theorem to S gives

$$\iint_S \mathbf{G}\cdot d\mathbf{S} = \iint_S \operatorname{curl}\mathbf{f}\cdot d\mathbf{S} = \oint_C \mathbf{F}\cdot d\mathbf{r}.$$

The same calculation applies to S' and gives exactly the same right-hand side.

c) Show that if G is superficially conservative, and S_0 is a closed surface, then

$$\iint_{S_0} \mathbf{G}\cdot d\mathbf{S} = 0.$$

In order to make sense of the integral, we're assuming that S_0 is oriented. Draw a small "circle" C (it won't be exactly flat, but that's OK) anywhere on S_0. The inside of C is a small disc D, and the outside of C (which is most of S_0) is a surface we can call S_1. Both D and S_1 inherit normal vectors (orientations) from S_0. Orient C as for Stokes' theorem on S_1, so that as you walk around C in the positive direction with S_1 on your left, your head points in the direction of the normal to S_1.

Now define D' to be the small disc with the *opposite* orientation from the one inherited from S_0. This makes C the boundary of D', oriented as for Stokes' theorem. The definition of surface integrals shows that

$$\iint_D \mathbf{G}\cdot d\mathbf{S} = -\iint_{D'} \mathbf{G}\cdot d\mathbf{S}$$

and

$$\iint_{S_0} \mathbf{G}\cdot d\mathbf{S} = \iint_{S_1} \mathbf{G}\cdot d\mathbf{S} + \iint_D \mathbf{G}\cdot d\mathbf{S}.$$

Therefore

$$\iint_{S_0} \mathbf{G}\cdot d\mathbf{S} = \iint_{S_1} \mathbf{G}\cdot d\mathbf{S} - \iint_{D'} \mathbf{G}\cdot d\mathbf{S}$$

On the other hand, S_1 and D' are two surfaces having the same boundary curve C; so the assumption that $\mathbf{G}$ is superficially conservative means that the two terms on the right cancel, giving zero as we wanted.

d) Show that if G is superficially conservative, and the first partial derivatives of its coordinates are continuous, then $\operatorname{div}\mathbf{G} = 0$.

Let B be any ball in space, and S its boundary sphere (oriented outward). By part (c), $\iint_S \mathbf{G}\cdot d\mathbf{S} = 0$. By the divergence theorem, the left side is $\iiint_B \operatorname{div}\mathbf{G}dV$. That is, the integral of the divergence of $\mathbf{G}$ over any ball must be zero. The only way that can happen is if the divergence is zero everywhere. (If for example it were positive at some point, then by continuity it would have to remain positive in a small ball around that point, and the integral over that ball would be positive.)

e) Suppose

$$\mathbf{G} = (y-z)\mathbf{i} + (z-x)\mathbf{j} + (x-y)\mathbf{k}.$$

Find a vector field F with the property that $\mathbf{G} = \mathbf{curl}\,\mathbf{F}$.

We need to find three functions M, N, and P (the coordinates of $\mathbf{F}$) satisfying

$$\frac{\partial P}{\partial y} - \frac{\partial N}{\partial z} = y - z,$$

and two more similar equations. What makes this hard is that the equations have too many solutions; there isn't an obvious way to pick one out. Staring at the equations for a while might suggest that you could try to arrange

$$\frac{\partial P}{\partial y} = y, \qquad \frac{\partial N}{\partial z} = z,$$

and four more similar equations. These six equations are actually easier to solve. It's even possible to write down the most general solution; but you only need one, like

$$M(x,y,z) = \frac{1}{2}(y^2+z^2), \qquad N(x,y,z) = \frac{1}{2}(x^2+z^2), \qquad P(x,y,z) = \frac{1}{2}(x^2+y^2).$$

Again, many other answers work. A different-looking one is

$$M(x,y,z) = 0, \qquad N(x,y,z) = x^2/2 - xy - yz, \qquad P(x,y,z) = x^2/2 - xz - yz.$$

2. (15 points) Let S be the cylinder

$$S = \{(x,y,z) \mid x^2+y^2 = 1, \quad 0 \le z \le 1\},$$

oriented with an outward-pointing normal. Let C_1 be the top circle

$$C_1 = \{(x,y,1) \mid x^2+y^2 = 1\},$$

oriented counterclockwise, and C_0 the bottom circle

$$C_0 = \{(x,y,0) \mid x^2+y^2 = 1\},$$

also oriented counterclockwise. Finally (to make a tin can) look at the top and bottom discs

$$T_1 = \{(x, y, 1) \mid x^2 + y^2 \le 1\},$$
$$T_0 = \{(x, y, 0) \mid x^2 + y^2 \le 1\}.$$

Orient T_1 with the upward normal $\mathbf{k}$, and T_0 with the downward normal $-\mathbf{k}$. Finally, let D be the inside of the can

$$D = \{(x, y, z) \mid x^2 + y^2 \le 1, \quad 0 \le z \le 1\}.$$

Consider the two vector fields

$$\mathbf{F} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k},$$

$$\mathbf{G} = (y - z)\mathbf{i} + (z - x)\mathbf{j} + (x - y)\mathbf{k}.$$

a) Calculate the curl and the divergence of F and G.

$$\operatorname{curl} \mathbf{F} = 0, \qquad \operatorname{curl} \mathbf{G} = -2\mathbf{i} - 2\mathbf{j} - 2\mathbf{k}.$$
$$\operatorname{div} \mathbf{F} = 3, \qquad \operatorname{div} \mathbf{G} = 0.$$

b) Calculate the flux of F, G, curl F, and curl G through each of the three surfaces S, T_0, and T_1.

Because S is a level surface for x^2+y^2, you can calculate a normal using the gradient of x^2+y^2. The outward unit normal to S at (x, y, z) is $\mathbf{n} = x\mathbf{i} + y\mathbf{j}$. Its dot product with $\mathbf{F}$ is $x^2 + y^2$, which is 1 on S; so

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \iint_S 1 \, dS = \text{area of } S = 2\pi.$$

Similarly, we calculate $\mathbf{G} \cdot \mathbf{n} = z(y - x)$. In cylindrical coordinates, since we're on the unit circle in the (x, y) plane, this is $z(\sin\theta - \cos\theta)$. When we integrate over S, we're first integrating around each horizontal circle. The integrals of $\sin\theta$ and $\cos\theta$ from 0 to 2π are zero, so we get

$$\iint_S \mathbf{G} \cdot d\mathbf{S} = 0.$$

c) Calculate the integrals of F, G, curl F, and curl G around each of the closed curves C_1 and C_2.

d) Calculate the integrals of div F and div G over the region D.

The base of D is a circle of radius 1, so its area is π. The height of D is 1, so the volume of D is π. Therefore

$$\iiint_D \operatorname{div} \mathbf{F} = 3\pi, \qquad \iiint_D \operatorname{div} \mathbf{G} = 0.$$

e) Stokes' theorem and the divergence theorem provide a lot of relationships among your various answers to parts (b), (c), and (d). Write down as many of them as you can think of, in the forms that would apply to any vector field F. (This means I am looking for formulas like

$$\iint_{T_0} \mathbf{curl}\,\mathbf{F} \cdot d\mathbf{S} = -\oint_{C_0} \mathbf{F} \cdot d\mathbf{r}$$

instead of formulas with your particular functions written out.)

3. (5 points). For the vector fields F and G of problem 2,

a) Find a function f with grad f = F. Is there another function f' with grad f' = G?

Since **F** has curl equal to zero, it must be the gradient of some f. We have to solve three equations

$$\frac{\partial f}{\partial x} = x, \qquad \frac{\partial f}{\partial y} = y, \qquad \frac{\partial f}{\partial z} = z.$$

The first gives $f(x, y, z) = x^2/2 + g(y, z)$; the second gives $g(y, z) = y^2/2 + h(z)$; and the third gives $h(z) = z^2/2 + c$. Therefore

$$f(x, y, z) = (x^2 + y^2 + z^2)/2 + c.$$

Since the curl of **G** is not zero, **G** cannot be a gradient.

b) Find a vector field H with curl H = G. Is there another vector field H′ with curl H′ = F?

You already found such an **H** in part 2(e). Since the divergence of **F** is not zero, part 2(a) shows that **F** cannot be a curl.