18.02A PROBLEM SET 12 SOLUTIONS

1. (10 points) This problem is about the unit ball B in $\mathbb{R}^3$:

$$B = \{(x, y, z) \mid x^2 + y^2 + z^2 \leq 1\}.$$

The ball is divided into a top half and a bottom half:

$$B^+ = \{(x, y, z) \mid x^2 + y^2 + z^2 \leq 1, z \geq 0\}, \quad B^- = \{(x, y, z) \mid x^2 + y^2 + z^2 \leq 1, z \leq 0\}.$$

The two half balls are separated by a disk E in the x-y plane. (The E stands for "equator.") The top surface of the ball is

$$S^+ = \{(x, y, z) \mid x^2 + y^2 + z^2 = 1, z \geq 0\},$$

and the bottom surface is

$$S^- = \{(x, y, z) \mid x^2 + y^2 + z^2 = 1, z \leq 0\}.$$

Orient each of the three surfaces S^-, T, and S^+ with a generally upward normal (positive z coordinate). Consider the vector field $\mathbf{F} = (z + 1)\mathbf{k}$.

a) Calculate the flux of F (upward) through each of the three surfaces S^-, E, and S^+.

The upward normal to E is $\mathbf{n} = \mathbf{k}$, so the flux is

$$\iint_E \mathbf{F} \cdot \mathbf{n} dS = \iint_E (z + 1) dS = \iint_E dS.$$

This is just the area of E, which is π.

The upward normal to S^+ is $\mathbf{n} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$, so the flux is

$$\iint_{S^+} \mathbf{F} \cdot \mathbf{n} dS = \iint_{S^+} (z^2 + z) dS.$$

Now S^+ is the graph of the function $z = \sqrt{1 - x^2 - y^2}$ over the unit disk in the (x, y)-plane. Using the formula

$$dS = \sqrt{1 + \left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2}\, dx\, dy,$$

and making a nasty little calculation, we find

$$\frac{dS = 1}{\sqrt{1 - x^2 - y^2} dx\, dy = (1/z) dx\, dy.}$$

The flux is therefore

$$\iint_E (z^2 + z)/z\, dx\, dy = \iint_E (z + 1) dx\, dy.$$

To evaluate this integral, we need to substitute the function $\sqrt{1 - x^2 - y^2}$ for z. It's possible to compute in a straightforward way, but it's easier to think geometrically. The integral is the sum of two terms. The integral of z over E is equal to the volume under the graph of the function z; that is, to the volume of the hemisphere, or $2\pi/3$. The integral of 1 over E is equal to the area of E, or π. So the total flux of $\mathbf{F}$ through S^+ is $5\pi/3$.

Finally, the upward normal through S^- is $-x\mathbf{i} - y\mathbf{j} - z\mathbf{k}$. The surface S^- is the graph of the function $z = -\sqrt{1 - x^2 - y^2}$, so almost the same calculation as above shows that the flux of $\mathbf{F}$ upward through S^- is

$$\iint_E (z+1)dx\,dy.$$

This time the integral of the negative function z is the negative of a volume, or $-2\pi/3$, so the flux is $\pi/3$.

b) Calculate the two integrals

$$\iiint_{B^+} \mathbf{div\,F}\,dV, \qquad \iiint_{B^-} \mathbf{div\,F}\,dV.$$

The divergence of $\mathbf{F}$ is the constant 1, so each integral is equal to the volume of a hemisphere, or $2\pi/3$.

c) Explain how to use the divergence theorem as a check on your answers to (a) and (b).

The boundary of B^+ is the union of S^+ (with its upward normal) and E (with the *opposite*, or downward normal. The divergence theorem therefore says

$$\iiint_{B^+} \operatorname{div}\mathbf{F}\,dV = \iint_{S^+} \mathbf{F}\cdot d\mathbf{S} - \iint_E \mathbf{F}\cdot d\mathbf{S}.$$

With the numbers computed above, this says

$$2\pi/3 = 5\pi/3 - \pi.$$

Similarly, the outward oriented boundary of B^- is E and S^- with the opposite normal, so

$$\iiint_{B^-} \operatorname{div}\mathbf{F}\,dV = -\iint_{S^-} \mathbf{F}\cdot d\mathbf{S} + \iint_E \mathbf{F}\cdot d\mathbf{S}.$$

The numbers above give

$$2\pi/3 = -\pi/3 + \pi.$$

2. (15 points) Suppose $u(x,y,z)$ is a function defined everywhere in space, and having at least two continuous derivatives. The *Laplacian of u* is by definition

$$L(u) = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}.$$

a) Suppose that D is a region in space bounded by a closed surface S. Show that the flux of the gradient of u across S is equal to the integral of the Laplacian of u across D: that is,

$$\iint_S \nabla u\cdot d\mathbf{S} = \iiint_D L(u)dV.$$

According to the divergence theorem, the flux of $\operatorname{grad} u$ across S is equal to the integral of $\operatorname{div} \operatorname{grad} u$ across D:

$$\iint_S \nabla u \cdot d\mathbf{S} = \iiint_D \nabla \cdot \nabla u dV.$$

But it is immediate from the definitions that the divergence of the gradient of u is equal to the Laplacian of u:

$$\nabla \cdot \nabla u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}.$$

b) Suppose now that we have found a function u with the property that the gradient of u on the unit sphere is equal to the outward unit normal to the sphere. Calculate the flux of ∇u outward through the unit sphere.

By hypothesis ∇u is equal to the outward unit normal, so $\nabla u \cdot \mathbf{n}$ is equal to 1, and the flux is equal to the surface area of the sphere, or 4π.

c) If u is a function with the property specified in (b), and B is the unit ball, show that

$$\iiint_B L(u) dV = 4\pi.$$

Explain why it follows that $L(u)$ must be at least as big as 3 somewhere in B.

The integral formula is parts (a) and (b) combined. If a function f is always smaller than M, then its integral is at most M times the volume of the region of integration:

$$\iiint_D f\, dV \leq M \operatorname{vol}(D).$$

The volume of the ball B is $4\pi/3$, so if $L(u)$ is strictly smaller than 3 everywhere in B, it would follow that

$$\iiint_B L(u) dV < 3 \cdot 4\pi/3 = 4\pi,$$

a contradiction. So $L(u)$ has to be at least 3 somewhere.

d) Find a specific function u with the property in (b), and find a specific place in B where $L(u)$ is at least 3. (Hint: part of what you want is that the gradient of u should be perpendicular to the surface $x^2 + y^2 + z^2 = 1$. What is the gradient of a function perpendicular to?)

The gradient of a function f is always perpendicular to the level surfaces $f = c$. Therefore the gradient of $v(x, y, z) = x^2 + y^2 + z^2$ is perpendicular to each sphere $x^2 + y^2 + z^2 = r^2$. In fact the gradient of v is $2x\mathbf{i} + 2y\mathbf{j} + 2z\mathbf{k}$, which has length $2\sqrt{x^2 + y^2 + z^2}$, which is 2 on the boundary sphere of the unit ball. So the gradient of v is twice the outward unit normal. We can fix that by taking

$$u(x, y, z) = (x^2 + y^2 + z^2)/2,$$

whose gradient is equal to the unit outward normal.

The Laplacian of u is easily calculated to be the constant 3; that is, any point of B will do.

You can make a slightly more interesting example with the function $u' = u^2$; it also has the property that the gradient on the unit sphere is equal to the outward unit normal, but

$$L(u') = 5(x^2 + y^2 + z^2),$$

or five times the square of the distance to the center. The Laplacian takes values from 0 to 5 on the ball; one place where its value is at least 3 is $(4/5, 0, 0)$ (where the value is $16/5$).

3. (10 points) A roller coaster is designed with the parametric equations

$$x(t) = \cos t, \quad y(t) = \sin t, \quad z(t) = 1 - \cos(3t),$$

The force of gravity on a roller coaster car is

$$\mathbf{F}(x, y, z) = -mg\mathbf{k},$$

with m the mass of the car and g the acceleration of gravity.

a) Let C_1 be the curve followed by the roller coaster from $t = 0$ to $t = \pi$. Calculate the work

$$\int_{C_1} \mathbf{F} \cdot d\mathbf{r}$$

done by gravity on the car in traversing this path.

We can use the given parametrization of C to compute the line integral. Since $\mathbf{F}$ has only a $\mathbf{k}$ coordinate, only dz matters:

$$dz = (\text{derivative of } 1 - \cos(3t))dt = 3\sin(3t)\, dt.$$

The line integral is therefore

$$\int_0^\pi (-mg) \cdot 3\sin(3t)\, dt = mg\cos(3t)]_0^\pi = mg(\cos(3\pi) - \cos(0)) = -2mg.$$

The answer is negative because the roller coaster goes from height 0 at time 0 to height 2 at time π: the car does work against gravity, instead of gravity doing work on the car.

The calculation is even easier if you notice that $\mathbf{F}$ is the gradient of the function $-mgz$, and use FTC.

b) Same question, but now for the path C_2 followed from $t = \pi$ to $t = 2\pi$.

The answer is zero, either by an easy repetition of the long calculation, or by noticing that the field is conservative and that the path is closed.

c) Now we'll move the roller coaster to a tiny asteroid, where the force of gravity is

$$\mathbf{E}(x, y, z) = -\frac{mg}{(z+1)^2}\mathbf{k}.$$

Calculate the work

$$\int_{C_1} \mathbf{E} \cdot d\mathbf{r}.$$

(Hint: if you write out the line integral and compute, this is a little bit unpleasant. There's an easier way.)

The easier was is to notice that $\mathbf{E}$ is the gradient of $mg/(z+1)$. The z coordinate increases from 0 to 2 over the path, so FTC says that the integral is

$$mg/(2+1) - mg/(0+1) = -2mg/3.$$

Even though gravity on the surface of the asteroid is the same as on the surface of the earth (since $\mathbf{E} = \mathbf{F}$ when $z = 0$), on the asteroid the gravity falls off rapidly, so less than half as much work is required to get to the top.

4. (10 points).

a) Suppose f is a differentiable function of one variable, and $\mathbf{F}(x, y, z) = f(z)\mathbf{k}$. Show that $\mathbf{F}$ must be conservative, and explain how to use f to find a potential function for $\mathbf{F}$.

You can use the curl condition for conservative: curl involves partial derivatives of each coordinate function with respect to the *other* coordinates, so it has to vanish for $\mathbf{F}$. (Only the z coordinate is non-zero, and its partial derivatives with respect to x and y are zero.)

Alternatively, the fact that $\mathbf{F}$ is conservative follows from the existence of a potential function. What you need to do is find an antiderivative g for f, such as

$$g(z) = \int_0^z f(t)\,dt.$$

Then the function $u(x, y, z) = g(z)$ is a potential for $\mathbf{F}$. Notice that this problem gives a way to solve the preceding problem, which is meant to remind you that life is unfair.

b) Suppose $\mathbf{E}$ is the vector field

$$\mathbf{E}(x, y, z) = \frac{1}{\rho}(x\mathbf{i} + y\mathbf{j} + z\mathbf{k}).$$

(Here $\rho(x, y, z) = \sqrt{x^2 + y^2 + z^2}$.) The vector field is not defined at the origin. Show that $\operatorname{curl}\mathbf{E} = 0$, and find a potential function for $\mathbf{E}$.

There are lots of straightforward ways to do this. We could calculate $\operatorname{curl}\mathbf{E}$ directly, but there is really no need since we are going to write $\mathbf{E}$ as the gradient of something (and then the vanishing of curl is automatic). So I'll just look for the potential function $f(x, y, z)$. One straightforward way to get that is to recall that

$$\frac{\partial f}{\partial x} = \frac{x}{\sqrt{x^2 + y^2 + z^2}}.$$

Getting an antiderivative for the right side in x leads to $f(x, y, z) = \sqrt{x^2 + y^2 + z^2} + g(y, z)$. Now looking at the y partial derivative shows that $g(y, z) = h(z)$, and finally looking at the z partial derivative shows that $h(z) = c$; so the only candidate potentials are $\sqrt{x^2 + y^2 + z^2} + c$. It's easy to check that these work.

Here is a way with less calculation. If C is any path that stays on the sphere of radius r, then $\mathbf{E}$ is always perpendicular to the path, so

$$\int_C \mathbf{E} \cdot d\mathbf{r} = 0.$$

It follows that the potential function has to be constant on each sphere of radius r. We only need to see how the potential depends on r. For that, we can fix some (non-zero) radius r_0, and calculate the work done in moving in a straight line C from $(r_0, 0)$ to $(r, 0, 0)$. Using x as a parameter makes the integral

$$\int_{r_0}^{r} \frac{x}{\sqrt{x^2 + 0^2 + 0^2}}\,dx = \int_{r_0}^{r} dx = r - r_0.$$

This shows that a potential function (if one exists) is

$$f(x, y, z) = r - r_0 = \sqrt{x^2 + y^2 + z^2} - r_0.$$

Again it's easy to check that $\nabla f = \mathbf{E}$.