18.02A PROBLEM SET 11 SOLUTIONS

1. (15 points) Let B be the unit ball in $\mathbb{R}^3$:

$$B = \{(x, y, z) \mid x^2 + y^2 + z^2 \leq 1\}.$$

This problem is about the function

$$f(x, y, z) = \sqrt{1 - x^2 - y^2 - z^2},$$

which is defined on B.

a) What are the largest and smallest values of the function f?

Since the square root function is by definition non-negative, the smallest value that f could possibly take would be zero. It does take that value, whenever $x^2 + y^2 + z^2 = 1$ (for example at the point $(1, 0, 0)$.) Since squares are non-negative, the largest value that $1 - x^2 - y^2 - z^2$ could possibly take is 1; and since the square root function is increasing, the largest value that f could possibly take is $\sqrt{1} = 1$. It does take that value, at the point $(0, 0, 0)$.

b) The volume of B is equal to $4\pi/3$. Using just this fact and the answer to (a), what conclusion can you draw about the triple integral $\iiint_B f\, dV$? (Your answer should be something like "the integral is at most 11.")

In one variable, the value of a definite integral is at least the minimum value of the function times the length of the interval, and at most the maximum value times the length of the interval. For double and triple integrals, the same rule applies, except that length is replaced by area or volume: the integral of f over a region in space is at least the minimum value times the volume of the region, and at most the maximum value times the volume of the region. In our case, we conclude

$$0 \leq \iiint_B f\, dV \leq 1 \cdot 4\pi/3.$$

c) There are at least three ways to calculate the integral: using rectangular coordinates, cylindrical coordinates, or spherical coordinates. Explain which of these ways ought to be the easiest.

One of the spherical coordinates is the distance ρ to the origin. Both the region and the function are easy to describe in terms of ρ: the region is $\{\rho \leq 1\}$, and the function is $\sqrt{1 - \rho^2}$. So spherical coordinates ought to be the easiest for this calculation.

d) Compute the integral in two different ways.

In spherical coordinates, the region is

$$0 \leq \rho \leq 1, \qquad 0 \leq \phi \leq \pi, \qquad 0 \leq \theta \leq 2\pi,$$

and I already wrote the function. Recalling that the volume element is $\rho^2 \sin\phi\, d\rho\, d\phi\, d\theta$, we find that the integral is

$$\int_0^{2\pi} \int_0^{\pi} \int_0^1 \sqrt{1 - \rho^2}\rho^2 \sin\phi\, d\rho\, d\phi\, d\theta.$$

The first integral is tough. To deal with the square root, it's a good idea to try a substitution like $\rho = \sin\tau$, $d\rho = \cos\tau\, d\tau$. As τ runs from 0 to $\pi/2$, $\sin\tau$ runs from 0 to 1. So the first integral (ignoring the constant $\sin\phi$ is

$$\int_0^1 \sqrt{1 - \rho^2}\rho^2\, d\rho = \int_0^{\pi/2} \sin^2\tau\, \cos^2\tau\, d\tau.$$

(One of the cosines comes from the formula $\cos\tau = \sqrt{1-\sin^2\tau}$, and the other from the formula for $d\rho$.) Now there are lots of ways to use half angle formulas to reduce the powers of sine and cosine and actually get to an antiderivative; but here is something a little quicker. Because $\sin 2\tau = 2\sin\tau\cos\tau$, the integrand is $(\cos^2 2\tau)/4$. It's a Handy Integration Fact that, over any interval of length a multiple of $\pi/2$, the average value of $\sin^2$ or $\cos^2$ is equal to $1/2$. In our case the range of values of 2τ is an interval of length π, so we can apply this fact. The conclusion is that the integral is equal to $(1/2)$ (the average value of $\cos^2 2\tau$) times $\pi/2$ (the length of the interval) times $1/4$ (a constant in the integrand):

$$int_0^1 \sqrt{1-\rho^2}\rho^2\, d\rho = \pi/16.$$

The next integral (putting back the $\sin\phi$) is

$$\int_0^\pi (\pi/16)\sin\phi\, d\phi.$$

This one is easy: it's $\pi/8$. The last integral gives

$$\int_0^{2\pi} \pi/8\, d\theta = \pi^2/4.$$

In rectangular coordinates, the region is

$$x^2+y^2+z^2 \le 1,$$

which (for fixed x and y) amounts to

$$z^2 \le 1-x^2-y^2,$$

or

$$-\sqrt{1-x^2-y^2} \le z \le \sqrt{1-x^2-y^2}.$$

Continuing in this way leads to

$$\int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}}\int_{-\sqrt{1-x^2-y^2}}^{\sqrt{1-x^2-y^2}} \sqrt{1-x^2-y^2-z^2}\, dz\, dy\, dx$$

The first integral

$$\int_{-\sqrt{1-x^2-y^2}}^{\sqrt{1-x^2-y^2}} \sqrt{1-x^2-y^2-z^2}\, dz$$

represents the area of a semicircle of radius $\sqrt{1-x^2-y^2}$; it is therefore equal to $\pi(1-x^2-y^2)/2$. The second integral is

$$\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \pi(1-x^2-y^2)/2\, dy.$$

This one is easy: we're just integrating constants and y^2, and the only confusion comes from the square roots in the limits. The answer is

$$(2\pi/3)(1-x^2)^{3/2}.$$

For the last integral, we need to integrate this with respect to x from -1 to 1. The square root suggests the substitution $x = \sin\tau$, with τ running from $-\pi/2$ to $\pi/2$ and $dx = \cos\tau\,d\tau$. The integral is therefore

$$(2\pi/3)\int_{-\pi/2}^{\pi/2}\cos^4\tau\,d\tau.$$

You can use the identity $\cos^2\tau = (\cos 2\tau + 1)/2$ to make the integrand into

$$(\cos^2 2\tau)/4 + (\cos 2\tau)/2 + 1/4.$$

According the Helpful Integration Hint above, the $\cos^2$ has average value 1/2, and an even easier Hint says that the cos has average value 0. So the integrand has average value $1/8 + 1/4 = 3/8$, and the result is

$$(2\pi/3)(3/8)(\pi) = \pi^2/4.$$

In cylindrical coordinates, plugging in r^2 for $x^2 + y^2$ in the equation for the region gives

$$r^2 + z^2 \leq 1, \qquad -\sqrt{1-r^2} \leq z \leq \sqrt{1-r^2}.$$

So the integral is

$$\int_0^{2\pi}\int_0^1\int_{-\sqrt{1-r^2}}^{\sqrt{1-r^2}}\sqrt{1-r^2-z^2}\,r\,dz\,dr\,d\theta.$$

The inner integral is r times the area of a semicircle of radius $\sqrt{1-r^2}$, or $\pi r(1-r^2)/2$. The next integral is $\pi/2$ times the integral from 0 to 1 of $r - r^3$; the value is $\pi/2$ times $1/2 - 1/4$, or $\pi/8$. The outer integral is just $\pi/8$ times the length of the interval, or $\pi^2/4$.

Notice that the answer I gave for part (c), that spherical coordinates "ought" to be the easiest, turned out to be mistaken: cylindrical coordinates are a bit easier, at least if you're clever enough to notice the geometric interpretation of the first integral.

e) Let R be the unit disk in $\mathbb{R}^2$:

$$A = \{(x,y) \mid x^2 + y^2 \leq 1\}.$$

What is the geometric meaning of the double integral $\iint_R \sqrt{1-x^2-y^2}\,dA$?

The graph of the function $\sqrt{1-x^2-y^2}$ is the hemisphere $z = \sqrt{1-x^2-y^2}$. The integral is therefore the volume under the hemisphere, or half the volume of the unit ball in three dimensions.

f) Can you imagine a geometric meaning for the triple integral $\iiint_B f\,dV$?

By analogy with (e), you might eventually imagine that the graph of f (in four dimensions!) cuts out half of a four-dimensional ball. The integral should be half of the volume of a four-dimensional ball. This is correct: your fancy geometry fact of the week says that

$$\text{volume of 4-dimensional ball of radius } r = \pi^2 r^4/2.$$

2. (10 points) This problem is about the tetrahedron

$$T = \{(x,y,z) \mid x \geq 0,\ y \geq 0,\ z \geq 0,\ x+y+z \leq 1\}.$$

You may assume the fact that the volume of T is $1/6$.

a) If $P = (a, b, c)$ is a fixed point and $Q = (x, y, z)$ is a point in T, then the square of the distance from P to Q is $(x-a)^2 + (y-b)^2 + (z-c)^2$. Find the average value of the squared distance from P to points of T. (The answer will depend on a, b, and c.)

To find the average value of a function on a region, you integrate it over the region and divide by the volume of the region. So the average squared distance we want is

$$6 \iiint_T ((x-a)^2 + (y-b)^2 + (z-c)^2)\, dV.$$

We need to make this into an iterated integral. For fixed x and y, z varies from 0 to $1 - x - y$ in T. (As usual, this comes from solving the equations describing T for z in terms of x and y.) Continuing in this way, we get

$$6 \int_0^1 \int_0^{1-x} \int_0^{1-x-y} ((x-a)^2 + (y-b)^2 + (z-c)^2)\, dz\, dy\, dx.$$

All the integration is of polynomials, so it's in principle easy. The three summands in the integral behave a bit differently, so let me first concentrate on the easiest one $(x-a)^2$. The inside integral with respect to z (ignoring the factor of 6) is

$$(1 - x - y)((x-a)^2 = (1-x)(x-a)^2 - y(x-a)^2.$$

The middle integral with respect to y gives (still without the 6)

$$(1-x)^2(x-a)^2 - ((1-x)^2/2)(x-a)^2 = ((1-x)^2/2)(x-a)^2.$$

The last integral (still without the factor of 6) is therefore

$$\int_0^1 (x^4/2 - (1+a)x^3 + (1/2 + 2a + a^2/2)x^2 - (a^2+a)x + a^2/2)\, dx.$$

This is

$$1/10 - (1+a)/4 + (1/2 + 2a + a^2/2)/3 - (a^2+a)/2 + a^2/2 = a^2/6 - a/12 + 1/60.$$

Putting back the factor of 6 gives

$$a^2 - a/2 + 1/10.$$

The next two summands could be treated similarly, but the calculations are a bit uglier when the iterated integral is written in this order. The easy thing is to say that by the symmetry of the problem in x, y, and z, they must yield the same formulas with b and c replacing a. Therefore the average squared distance is

$$(a^2 - a/2 + 1/10) + (b^2 - b/2 + 1/10) + (c^2 - c/2 + 1/10).$$

b) What point $P = (a, b, c)$ minimizes the average value of the squared distance to points of T?

To minimize the sum of a function of a, a function of b, and a function of c, you should minimize each summand separately. This is almost trivial: the unique minimum is at $a = b = c = 1/4$.

c) Find the center of mass of the tetrahedron.

This is worked out in the text on pages 733–734; the answer is $(1/4, 1/4, 1/4)$.

d) Can you make any insightful comments suggested by this problem?

What I hope you would guess is that the centroid of any solid minimizes the average value of the square of the distance. More generally (for solids of varying density) the moment of inertia about a point is minimal when the point is the center of mass of the solid.