18.02A PROBLEM SET 10 SOLUTIONS

1. (10 points) A function $f(x, y)$ of two variables is called *harmonic* if

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 0.$$

Any linear function $ax + by + c$ is harmonic, because all its second derivatives are zero.

a) Give an example of a harmonic function that isn't linear.

It isn't easy to solve this differential equation in general, but you're only asked to give an example of a solution. Since I mentioned linear functions already, one natural thing to try is the next most complicated functions: the quadratic functions. We might as well not have any linear or constant terms (since they won't affect whether we get a solution); so we're looking at a function like

$$h(x, y) = ax^2 + bxy + cy^2,$$

where a, b, and c are constants to be picked later. Then we calculate

$$\frac{\partial^2 h}{\partial x^2} + \frac{\partial^2 h}{\partial y^2} = 2a + 2c.$$

This will be zero whenever $a + c = 0$: for example, for the functions $h_1(x, y) = x^2 - y^2$ or $h_2(x, y) = xy$. It's also possible to make up more complicated examples, like

$$h_3(x, y) = e^x \cos y, \qquad h_4(x, y) = e^x \sin y.$$

b) Suppose that f is a harmonic function (defined on the whole plane). Prove that the flux of ∇f through any closed curve is equal to zero.

By definition, the gradient ∇f is $\frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j}$. It follows that

$$\operatorname{div}\nabla f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 0.$$

That is, the vector field ∇f has divergence zero. According to the normal form of Green's theorem, the flux of ∇f out of any closed curve is equal to the integral of the divergence of ∇f (which is zero) over the region bounded by the curve.

c) Still supposing that f is harmonic, prove that the vector field

$$\mathbf{F} = -\frac{\partial f}{\partial y}\mathbf{i} + \frac{\partial f}{\partial x}\mathbf{j}$$

is conservative.

The integral of $\mathbf{F}$ over a closed curve C is equal to

$$\oint_C \mathbf{F} \cdot d\mathbf{R} = \oint_C -\frac{\partial f}{\partial y} dx + \frac{\partial f}{\partial x} dy.$$

That is exactly the formula for the flux of ∇f across C, so it vanishes by part (b). This means that $\mathbf{F}$ is conservative.

d) Now suppose f is your harmonic function from (a). Find a potential function $g(x, y)$ for the conservative field F of part (c).

(Going from f to g is always possible, and the new function g is always harmonic too. If you repeat the procedure (starting with the new harmonic function g in place of f) you'll get $-f$ plus a constant. Actually computing g in general is a subtle business.

I'll do one of the harder examples: $f = h_3 = e^x \cos y$. The vector field $\mathbf{F}$ is therefore

$$-\frac{\partial f}{\partial y}\mathbf{i} + \frac{\partial f}{\partial x}\mathbf{j} = e^x \sin y\mathbf{i} + e^x \cos y\mathbf{j}.$$

A potential function g is supposed to satisfy

$$\frac{\partial g}{\partial x} = e^x \sin y, \qquad \frac{\partial g}{\partial y} = e^x \cos y.$$

The first equation implies that

$$g(x, y) = e^x \sin y + a(y),$$

and then the second implies that $da/dy = 0$, so that a is a constant. Therefore

$$g(x, y) = e^x \sin y$$

plus a constant. This is the harmonic function h_4 from my list of examples in part (a).

2. **(10 points) This problem is about moving out. A vector field F in the plane (possibly not defined at the origin) is called *radial* if its direction at every point is parallel to the radius. The most general radial vector field looks like**

$$\mathbf{F}(x, y) = f(r)x\mathbf{i} + f(r)y\mathbf{j},$$

with f a function of the radius $r = \sqrt{x^2 + y^2}$.

a) What is the flux of a radial vector field outward through the circle of radius r centered at the origin? (The answer depends on $f(r)$.)

The normal vector to the circle always points out, parallel to $\mathbf{F}$; so the flux integrand is a constant, namely the length of $\mathbf{F}$ on the circle (with a minus sign if f is negative.) So we are integrating the constant $rf(r)$ over a circle of length $2\pi r$; the flux is $2\pi r^2 f(r)$.

b) The coordinates of F are

$$M(x, y) = f(r)x, \qquad N(x, y) = f(r)y.$$

Find a formula for div F as a function of r. (You'll need to use the chain rule to differentiate $f(r)$ with respect to x and y. Your answer should depend on f and f'.)

We calculate

$$\begin{aligned}\frac{\partial M}{\partial x} &= \frac{\partial f}{\partial x}x + f(r)\\ &= xf'(r)\frac{\partial r}{\partial x} + f(r).\end{aligned}$$

Since $r = \sqrt{x^2 + y^2}$, we calculate

$$\frac{\partial r}{\partial x} = (2x)(1/2)(x^2 + y^2)^{-1/2} = x/r.$$

Plugging in above gives

$$\frac{\partial M}{\partial x} = (x^2/r)f'(r) + f(r).$$

Similarly

$$\frac{\partial N}{\partial y} = (y^2/r)f'(r) + f(r).$$

Adding these two formulas gives

$$\operatorname{div}\mathbf{F} = ((x^2 + y^2)/r)f'(r) + 2f(r) = rf'(r) + 2f(r).$$

c) Find a non-zero function $f(r)$ with the property that $f(1) = 1$, but

$$\mathbf{div}\,\mathbf{F} = 0.$$

We need to find a non-zero function f so that

$$rf'(r) + 2f(r) = 0.$$

You can solve this differential equation by separating variables. Alternatively, you can remember the rule for differentiating powers, and guess that some function $f(r) = r^m$ might work. This leads quickly to $m = -2$: that is

$$f(r) = r^{-2}.$$

d) For the vector field in part (c), your answer to (a) should say that the flux of F out of the unit circle is equal to 2π. Calculating that flux with the normal form of Green's theorem seems to give 0. Explain this apparent contradiction.

The vector field is

$$\mathbf{F}(x, y) = \frac{x}{x^2 + y^2}\mathbf{i} + \frac{y}{x^2 + y^2}\mathbf{j}.$$

Its divergence is zero everywhere it's defined, but it is not defined at $x = y = 0$. We can't apply Green's theorem to the unit circle, because the vector field isn't defined on the whole inside of the circle.

3. (5 points) Problem 21.3(18) on page 770.

Since $\genfrac{}{}{0pt}{}{\text{this was a straightforward calculation}}{\text{I am a lazy bum}}$ I won't write out a solution.