18.01 Problem Set 7 Solutions

Part II: 15 points

1) The bell-shaped curve is the graph of the function

\[ f(t) = \frac{1}{\sqrt{2\pi}} e^{-t^2/2}. \]

The total area under this curve turns out to be 1 (this is a hard fact that we’ll see later in the semester). The area under the curve between 0 and \( x \) turns out to be the probability that a (normally distributed random variable) is at most \( x \) standard deviations above the mean. This function of \( x \) is very important in probability and statistics.

a) Sketch the graph of \( f \).

b) Use Simpson’s rule with two subintervals to estimate \( \int_{0}^{1} f(t) \, dt \); that is, the probability that a normal random variable is between the mean and one standard deviation above it. You can use a calculator to compute things like \( \sqrt{2\pi} \) and \( e^{-1/8} \) that come up, but keep at least six decimal places of accuracy.

The interval size is \( h = 1/2 \), and the Simpson’s rule approximation to the integral is

\[ \frac{h}{3} (f(0) + 4f(1/2) + f(1)) = \frac{1}{6\sqrt{2\pi}} \left( e^0 + 4e^{-1/8} + e^{-1/2} \right) \]

This is

\[ (.06649038) (1.00000000 + 4 \times (.88249690) + .60653066) = (.06649038)(5.13651827) = .34152905 \]

c) The fourth derivative of \( f \) is between 1.2 and –.5 on the interval from 0 to 1. According to the discussion on page 373 of Simmons, how big might the error in this calculation be? What the text says is that the error in Simpson’s rule is at most \( M \cdot 1 \cdot h^4/180 \), where \( M \) is
the largest value of the fourth derivative of \( f \). Since \( M \) is at most 1.2, the error should be at most \((1.2)/(180 \cdot 2^4) = .0004166\ldots\). That is, the true answer should be between .3411 and .3419.

d) Find a source on the web, or in a table in a book, that tells you the exact answer to (b) to at least six decimal places. (You should record here both the answer and your source for it.) What was the actual error in your calculation in (b)?

There is a java applet at http://davidmlane.com/hyperstat/z_table.html which gives the value .341345. So the error in the Simpson’s rule approximation was actually .000184: about a third of the bound appearing in the textbook. The reason for this is that the error in Simpson’s rule is actually equal to \( f^{(4)}(c)(b - a)h^4/180 \) for some value of \( c \) between \( a \) and \( b \). In theorems like this (which is like MVT), in reasonable examples, you expect the mysterious \( c \) to be nearer the middle of the interval than the ends. So it’s reasonable to guess that the error should be given by some value of the fourth derivative closer to .4 (the middle of the given range 1.2 to −.5). The actual value of the fourth derivative in this error formula is about .53.

2a) Sketch the graph of the function \( y = \sin^2 x \) over the interval \( 0 \leq x \leq \pi \). What does the picture suggest about the average value of \( \sin^2 x \) on this interval?

![Graph of \( y = \sin^2 x \)](image)

The picture looks symmetric about the line \( y = 1/2 \); the hump above that line can be divided vertically into two pieces which look exactly like the pieces below the line. This suggests that the average value should be equal to 1/2.

b) Calculate the average value of \( \sin^2 x \) over the interval \( 0 \leq x \leq \pi \). (One way to find an antiderivative of \( \sin^2 x \) is using the formula

\[
\sin^2 x = (1 - \cos(2x))/2.
\]

By definition the average value is

\[
\frac{1}{\pi} \int_0^\pi \sin^2(x)dx = \frac{1}{2\pi} \int_0^\pi (1 - \cos(2x))dx.
\]
The antiderivative in the last integral is \( x - \sin(2x)/2 \), so the average value is

\[
\frac{1}{2\pi} \left[ x - \sin(2x)/2 \right]_0^\pi = \frac{1}{2\pi} \left[ \pi - \sin(2\pi) - 0 + \sin(2 \cdot 0) \right] = 1/2.
\]

3) There’s an electrical engineering problem with Example 1 in section AV of the notes. What is it?

It’s stated that the amplitude of a 120 volt household AC circuit is 120 volts. That’s not correct. The amplitude is \( \sqrt{2} \cdot 120 \approx 170 \) volts. Here’s the reason. The most basic thing we want from an electrical circuit is energy; to make a light bulb shine, for example. A voltage of \( V \) acting against a resistance of \( R \) delivers a current (rate of flow of electrons) of \( I = V/R \), and power (rate at which work is done; the product of the current and the voltage barrier being overcome)

\[ P = V^2/R. \]

This power number \( P \) is measured in watts.

In an alternating circuit, the voltage amplitude is given by a sin function, going up and down. So the power being delivered is given by \( \sin^2 \): varying between 0 and the square of the maximum voltage. The designers of AC circuits wanted the reported voltage to reflect how much power the circuit could deliver; that is, the average value of this \( \sin^2 \) function. That’s half of the square of the maximum voltage. So they designed the maximum voltage to be \( \sqrt{2} \) times the reported 120, so that the average power delivery would be the same as with a 120 volt DC circuit.