1. (10 points) This problem concerns the function \( f(x) = \sqrt{1-x^2} \) on the interval \( 0 \leq x \leq 1 \). I want to approximate this slightly complicated function by a straight line \( y = ax + b \), where \( a \) and \( b \) are to be chosen. Usually when we talk about linear approximations in calculus, we are interested in \( x \) close to one special value \( x_0 \); but here I am considering all values of \( x \) equally. Since the function \( f(x) \) is strictly decreasing, it’s reasonable to think that a good approximation should be decreasing as well; so I will assume

\[ a \text{ is strictly negative.} \]

a) Show that \( f \) is concave down on the interval \( 0 < x < 1 \).

The derivative of \( f \) is \( f' = \frac{x}{\sqrt{1-x^2}} \), which is always negative. The numerator of \( f' \) is growing in absolute value with \( x \), and the denominator is decreasing; so \( f' \) grows in absolute value, which is to say that \( f' \) is decreasing. Therefore \( f \) is concave down.

b) Show that the derivative \( f'(x) \) decreases from 0 toward \(-\infty\) as \( x \) increases from 0 to 1.

Essentially I explained this in the answer for (a); the formula shows that \( f'(0) = 0 \), and as \( x \) approaches 1 the numerator in \( f' \) is decreasing to \(-1\) and the denominator is decreasing to 0, so \( f' \) decreases toward \(-\infty\).

c) Show that the “error function”

\[ e(x) = f(x) - ax - b \]

has exactly one critical point between 0 and 1, and that this critical point is a maximum.

The derivative is \( e'(x) = f'(x) - a \), which (by (b)) decreases from the positive number \(-a\) at zero toward \(-\infty\); so it passes through zero exactly once, when

\[ \frac{x}{\sqrt{1-x^2}} = -a. \]

Squaring both sides and solving gives

\[ x = \frac{-a}{\sqrt{1+a^2}}. \]

Since \( f \) is concave down, its second derivative is negative. The function \( e \) has exactly the same second derivative as \( f \), so it is concave down as well; so the critical point must be a maximum.

d) Show that the value of \( e \) at the critical point is \( \sqrt{1+a^2} - b \).

Plugging the value for \( x \) into the definition of \( e \) gives a value of

\[
\begin{align*}
&\sqrt{1 - \frac{a^2}{1+a^2} + \frac{a^2}{\sqrt{1+a^2}}} - b \\
= &\sqrt{\frac{(1+a^2) - a^2}{1+a^2} + \frac{a^2}{\sqrt{1+a^2}}} - b \\
= &\sqrt{\frac{1}{1+a^2} + \frac{a^2}{\sqrt{1+a^2}}} - b \\
= &\frac{1+a^2}{\sqrt{1+a^2}} - b = \sqrt{1+a^2} - b.
\end{align*}
\]

e) One way to choose a “best” straight line approximation to \( f \) is to require that the maximum value of \( |e(x)| \) be as small as possible. Explain why this maximum value is the largest of the three numbers

\[ |1-b|, \quad |\sqrt{1+a^2} - b|, \quad |-a-b|. \]
The maximum value of \( e(x) \) occurs either at the critical point or at one of the endpoints \( x = 0 \) or \( x = 1 \). For the same reason, the maximum value of \(-e(x)\) occurs either at the critical point or at the endpoints. The first and last numbers on the list are \( |e(0)| \) and \( |e(1)| \), and the middle one is the absolute value at the critical point (computed in (d)).

f) **Show that if \(-1 \leq a \leq 0\), then the maximum value of \(|e|\) is \((\sqrt{1 + a^2} + a)/2\), and find a similar formula for the case \(-\infty < a \leq -1\).**

According to (e), we want to choose \( b \) to minimize the maximum distance from \( b \) to the three positive numbers 1, \( \sqrt{1 + a^2} \), and \(-a\). This is a problem similar in character to the chess club problem from the previous problem set. This time the best choice of \( b \) is halfway between the largest and smallest of the three numbers.

If \(-a\) is between 0 and 1, then it is the smallest of the three numbers, and \( \sqrt{1 + a^2} \) is the largest. The point halfway between is \( b = (\sqrt{1 + a^2} - a)/2 \), and its distance to \(-a\) is

\[
(\sqrt{1 + a^2} + a)/2.
\]

If \(-a\) is greater than 1, then the smallest number is 1 and the largest is \( \sqrt{1 + a^2} \), so the best choice for \( b \) is

\[
b = \frac{\sqrt{1 + a^2} + 1}{2}.
\]

Its distance to 1 (the maximum error) is

\[
\frac{\sqrt{1 + a^2} - 1}{2}.
\]

g) **What is the best straight line approximation to \( f \)? What is the maximum error of this approximation?**

For a fixed choice \( a < 0 \) of the line slope, we calculated the maximum error \( M(a) \) in (f):

\[
M(a) = \begin{cases} 
(\sqrt{1 + a^2} + a)/2 & \text{for } -1 \leq -a < 0 \\
(\sqrt{1 + a^2} - 1)/2 & \text{for } -\infty < a \leq -1.
\end{cases}
\]

So we must find the minimum value of the function \( M \). As \( a \) approaches 0, \( M(a) \) approaches 1/2. As \( a \) goes from 1 to \(-\infty\), \( M(a) \) (which is close to \(-a/2\)) increases from \((\sqrt{2} - 1)/2\) to \(+\infty\). Finally, \( M \) is continuous, and differentiable except where the formula changes at \( a = -1 \).

From these facts we conclude that the minimum exists, and that it occurs either at \( a = -1 \) or at a critical point \(-1 < a < 0 \). On this last interval,

\[
M'(a) = \frac{\sqrt{1 + a^2} + a}{2\sqrt{1 + a^2}}.
\]

The numerator is positive (since \( 1 + a^2 > a^2 \)), and the denominator is positive; so \( M'(a) > 0 \), and there are no critical points.

The best approximation is with \( a = -1 \) and \( b = (\sqrt{2} + 1)/2 \) (calculated in (f)):

\[
y = -x + \frac{1 + \sqrt{2}}{2},
\]

and the maximum error is \((\sqrt{2} - 1)/2\) (which occurs at \( x = 0, x = 1 \), and \( x = 1/\sqrt{2} \)).

2. **(5 points) Attached to every differentiation formula there is an integration formula: for example**

\[
d(u^n) = nu^{n-1}du \quad \leftrightarrow \quad \int nu^{n-1}du = u^n.
\]
It’s handy to adjust the constants on the right to make the integration formula more convenient, replacing \( n \) by \( n+1 \) and then dividing by \( n \):

\[
d(u^n) = nu^{n-1}du \quad \leftrightarrow \quad \int u^n du = u^{n+1}/(n+1) \quad (n \neq -1).
\]

If we do this with the inverse trig functions \( \sin^{-1} \) and \( \cos^{-1} \) we get two formulas

\[
d(\sin^{-1} u) = du/\sqrt{1-u^2} \quad \leftrightarrow \quad \int du/\sqrt{1-u^2} = \sin^{-1}(u). \quad (S)
\]

\[
d(\cos^{-1} u) = -du/\sqrt{1-u^2} \quad \leftrightarrow \quad \int du/\sqrt{1-u^2} = -\cos^{-1}(u). \quad (C)
\]

The trouble is that this leads to two completely different formulas \((S)\) and \((C)\) for the same antiderivative. Explain this as carefully and completely as you can: for example, if only one of these formulas is right, say which one, and what’s wrong with the other.

Defining the inverse trig functions requires some care and some choices. The definition on page 313 of Simmons says that

\[-\pi/2 \leq \sin^{-1}(x) \leq \pi/2, \quad \sin(\sin^{-1}(x)) = x \quad -1 \leq x \leq 1.\]

I couldn’t find inverse cosine in Simmons, but the standard definition has

\[0 \leq \cos^{-1}(x) \leq \pi, \quad \cos(\cos^{-1}(x)) = x \quad -1 \leq x \leq 1.\]

There is a standard trig identity

\[
\cos(\theta) = \sin(\pi/2 - \theta).
\]

If you combine this identity with the definitions of the inverse functions above, you find that

\[
\sin^{-1}(x) = \pi/2 - \cos^{-1}(x) \quad -1 \leq x \leq 1. \quad (E)
\]

Antiderivatives (indefinite integrals) are defined only up to an additive constant. What the explanation \((E)\) shows is that the antiderivative formula in \((C)\) is just the one in \((S)\) plus the constant \( \pi/2 \). They’re both correct.