May 2, 2011

18.01 Problem Set 11 solutions

Part II: 15 points

1. (10 points) This problem is about the functions $x^me^{-x}$, with $m$ a non-negative integer.

1a) Calculate the average value $A_0$ of $e^{-x}$ over the interval $[0,1]$.

By definition this is

$$
\int_0^1 e^{-x} \, dx = [-e^{-x}]_0^1 = -e^{-1} + e^0 = 1 - e^{-1} \approx .632
$$

b) Calculate the average value $A_1$ of $xe^{-x}$ over the interval $[0,1]$.

By definition this is $\int_0^1 xe^{-x} \, dx$. This is a good candidate for integration by parts, with $u = x$ (since the derivative of $x$ is simpler than $x$) and $dv = e^{-x} \, dx$ (since that’s easy to integrate). This gives $du = dx$, $v = -e^{-x}$, so

$$
\int_0^1 xe^{-x} \, dx = \left[ -xe^{-x} \right]_0^1 + \int_0^1 e^{-x} \, dx = -e^{-1} + \int_0^1 e^{-x} \, dx.
$$

The last integral we calculated in (a) was $A_0 = 1 - e^{-1}$, so the average value is

$$
A_1 = -e^{-1} + A_0 = 1 - 2e^{-1} \approx .264.
$$

c) Calculate the average value $A_2$ of $x^2e^{-x}$ over the interval $[0,1]$.

By definition this is $\int_0^1 x^2e^{-x} \, dx$. Again we can integrate by parts, with $u = x^2$ (since the derivative of $x^2$ is simpler than $x^2$) and $dv = e^{-x} \, dx$. We find $du = 2xdx$, $v = -e^{-x}$, and

$$
\int_0^1 x^2e^{-x} \, dx = \left[ -x^2e^{-x} \right]_0^1 + 2 \int_0^1 e^{-x} \, dx = -e^{-1} + 2A_1.
$$

Plugging in the value of $A_1$ from (b) gives

$$
A_2 = 2 - 5e^{-1} \approx .1606.
$$

d) Prove a reduction formula of the form

$$
\int x^m e^{-x} \, dx = C_m x^m e^{-x} + D_m \int x^{m-1} e^{-x} \, dx.
$$

Integrate by parts: use $u = x^m$, $dv = e^{-x} \, dx$, so that $du = mx^{m-1} \, dx$ and $v = -e^{-x}$. The formula is

$$
\int x^m e^{-x} \, dx = -x^m e^{-x} + m \int x^{m-1} e^{-x} \, dx.
$$

e) Explain how to calculate the average value $A_m$ of $x^me^{-x}$ over $[0,1]$ from $A_{m-1}$.

Applying the reduction formula gives

$$
\int_0^1 x^m e^{-x} \, dx = -x^m e^{-x} \bigg|_0^1 + m \int_0^1 x^{m-1} e^{-x} \, dx = -e^{-1} + m \int_0^1 x^{m-1} e^{-x} \, dx.
$$
In terms of the average values we are interested in, this says
\[ A_m = mA_{m-1} - e^{-1}. \]

f) Show that there are integers \( a_m \) and \( b_m \) with the property that
\[ A_m = a_m - \frac{b_m}{e}. \]

**Explain how to calculate \( a_m \) and \( b_m \) from \( a_{m-1} \) and \( b_{m-1} \).**

Certainly \( A_0 \) and \( A_1 \) and \( A_2 \) are shaped like this. We proceed by induction: suppose that \( A_{m-1} = a_{m-1} - \frac{b_{m-1}}{e} \), with \( a_{m-1} \) and \( b_{m-1} \) (positive) integers. According to (e),
\[ A_m = mA_{m-1} - \frac{1}{e} = ma_{m-1} - \frac{mb_{m-1} + 1}{e}. \]
This answer has the shape that we want, with
\[ a_m = ma_{m-1}, \quad b_m = mb_{m-1} + 1. \]
That’s all you needed to say. In fact you can easily calculate \( a_m \) and \( b_m \) using these formulas and the starting conditions (from (a))
\[ a_0 = 1, \quad b_0 = 1. \]
The answers are
\[ a_m = m!, \quad b_m = m! + m(m-1) \cdot \cdots \cdot 2 + m(m-1) \cdots 3 + \cdots + m(m-1) + m + 1. \]

g) **Explain why \( A_m \) is between \( \frac{1}{(m+1)!} \) and \( \frac{1}{e(m+1)!} \).**

What makes this question hard is thinking that it has something to do with parts (a)–(f). On the interval from 0 to 1, \( e^{-x} \) decreases from 1 to \( e^{-1} \). The function \( x^m e^{-x} \) is therefore always between \( x^m \) and \( x^m / e \). The average values therefore satisfy
\[ \text{(average value of } x^m \text{)} \geq A_m \geq \text{(average value of } x^m / e \text{)}. \]
The first average value is
\[ \int_0^1 x^m dx = \left[ \frac{x^{m+1}}{m+1} \right]_0^1 = \frac{1}{m+1}, \]
and the last is \( \frac{1}{e(m+1)!} \).

2. **(5 points) Explain why \( e - (1 + \frac{1}{1!} + \frac{1}{2!} + \cdots + \frac{1}{m!}) \) is between \( \frac{1}{(m+1)!} \) and \( \frac{e}{(m+1)!} \).** (This means, for instance, that the error in the approximation
\[ e \approx 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{1}{15!} \]
is at most \( e/15! \approx 2 \times 10^{-12}. \)) Parts (f) and (g) of Problem 1 (together with the formulas written there for \( a_m \) and \( b_m \)) say that
\[ \frac{1}{e(m+1)!} \leq \frac{m! + m(m-1) \cdot \cdots \cdot 2 + m(m-1) \cdots 3 + \cdots + m(m-1) + m + 1}{e} \leq \frac{1}{m+1}. \]
Multiplying this by \( e/m! \) gives
\[ \frac{1}{(m+1)!} \leq \left[ e - (1 + \frac{1}{1!} + \frac{1}{2!} + \cdots + \frac{1}{(m-1)!} + \frac{1}{m!}) \right] \leq \frac{e}{(m+1)!}. \]
Actually the first formula is interesting if you just multiply it by \( e \) it says
\[ \frac{1}{m+1} \leq e \cdot m! - [m! + m(m-1) \cdot \cdots \cdot 2 + m(m-1) \cdots 3 + \cdots + m(m-1) + m + 1] \leq \frac{e}{m+1}. \]
The expression in square brackets is an integer, so these inequalities (for \( m+1 > e \) tell you that \( e \cdot m! \) is not an integer: the reason is that it differs from an integer by something positive but less than 1. Since \( e \cdot m! \) can never be an integer (for any big \( m \)), it follows that \( e \) must be an irrational number.