April 15, 2011

18.01 Problem Set 11
Due Wednesday, April 27, in recitation

Collaboration and discussion of problem sets is a good idea; you must write up your answers on your own, and you must answer question 0 of Part II.

Part I: 10 points

Notation for homework problems: “2.4/13” means Problem 13 at the end of section 2.4 in Simmons. “1A-3” means Exercise 1A-3 in Section E (Exercises) of the Supplementary Notes.

1. 5F-1, 2dc, 3, 4.
2. 4E-1, 3, 7; 4F-1b, 7, 8; 4G-1, 3, 7.

Part II: 15 points

0. Write the names of all the people you consulted or with whom you collaborated and the resources you used, beyond the course text and notes and your instructors; or say “none” or “no consultation.”

1. (10 points) This problem is about the functions \(x^me^{-x}\), with \(m\) a non-negative integer.
   a) Calculate the average value \(A_0\) of \(e^{-x}\) over the interval \([0, 1]\).
   b) Calculate the average value \(A_1\) of \(xe^{-x}\) over the interval \([0, 1]\).
   c) Calculate the average value \(A_2\) of \(x^2e^{-x}\) over the interval \([0, 1]\).
   d) Prove a reduction formula of the form
      \[
      \int x^me^{-x}dx = C_m x^m e^{-x} + D_m \int x^{m-1}e^{-x}dx.
      \]
   e) Explain how to calculate the average value \(A_m\) of \(x^me^{-x}\) over \([0, 1]\) from \(A_{m-1}\).
   f) Show that there are integers \(a_m\) and \(b_m\) with the property that
      \[
      A_m = a_m - \frac{b_m}{e}.
      \]
      Explain how to calculate \(a_m\) and \(b_m\) from \(a_{m-1}\) and \(b_{m-1}\).
   g) Explain why \(A_m\) is between \(\frac{1}{(m+1)!}\) and \(\frac{1}{e(m+1)!}\).

2. (5 points) Explain why \(e - (1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{1}{m!})\) is between \(\frac{1}{(m+1)!}\) and \(\frac{e}{(m+1)!}\). (This means, for instance, that the error in the approximation
   \[
   e \approx 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{1}{15!}
   \]
   is at most \(e/15! \approx 2 \times 10^{-12}\).