1. [3]

(a) By Chebyshev’s inequality we have: \( \Pr(X = 0) \leq \Pr(|X - \mathbb{E}X| \geq \mathbb{E}X) \leq \text{Var}(X)/(\mathbb{E}X)^2 \).

(b) Indeed

\[
\text{Var}(X) = \mathbb{E}(X - \mathbb{E}X)^2 \\
= \mathbb{E}((\sum_i (X_i - \mathbb{E}X_i))^2) \\
= \sum_i \sum_j \mathbb{E}((X_i - \mathbb{E}X_i)(X_j - \mathbb{E}X_j)) \\
= \sum_i \sum_j \text{Cov}(X_i, X_j) \\
= \sum_{i\neq j} \text{Cov}(X_i, X_j)
\]

where \( \text{Cov}(X, Y) := \mathbb{E}(XY) - \mathbb{E}X\mathbb{E}Y \). Note \( \text{Cov}(X_i, X_j) = 0 \) if \( i \neq j \). We have \( \text{Cov}(X_i, X_j) \leq \mathbb{E}(X_iX_j) = \Pr(E_i \cap E_j) \) and \( \text{Cov}(X_i, X_i) \leq \mathbb{E}(X_i^2) = \mathbb{E}X_i \) so

\[
\text{Var}(X) \leq \sum_i \mathbb{E}X_i + \sum_{i\neq j} \Pr(E_i \cap E_j) \\
= \mathbb{E}X + \sum_i \Pr(E_i) \sum_{j \neq i} \Pr(E_j | E_i)
\]

(c) We mimic the proof given in class for \( H = P_{t+1} \). Note that containing \( H \) is the same as \( X \geq 1 \). For any \( v \)-subset \( S \) of \( V \), we have \( p^v \leq \mathbb{E}X_S = \Pr(E_S) \leq v!p^v \), since there are at most \( v! \) possible copies of \( H \) in \( S \). Thus \( \mathbb{E}X_S = \Pr(E_S) = \Theta(p^v) \) as \( n \to \infty \), since \( v \) is a constant. We also have \( \mathbb{E}X = \sum_S \mathbb{E}X_S = \binom{n}{v} \mathbb{E}X_S = \Theta(n^vp^v) \). (Note \( (n/v)^v \leq \binom{n}{v} \leq n^v \) so \( \binom{n}{v} = \Theta(n^v) \).) Now if \( p = o(t) = o(n^{-v/c}) \), \( \Pr(X \geq 1) \leq \mathbb{E}X = O(n^vp^v) = o(1) \), by Markov’s inequality.
We now assume $p = \omega(t) = \omega(n^{-v/e})$. We have $\Pr(X = 0) \leq \Pr(|X - \mathbb{E}X| \geq \mathbb{E}X) \leq \text{Var}(X)/\mathbb{E}X^2$ by Chebyshev’s inequality. We want to show that $\Pr(X \geq 1) = 1 - o(1)$ or $\Pr(X = 0) = o(1)$.

Thus we want to show $\text{Var}(X)/\mathbb{E}X^2 = o(1)$.

Since $\mathbb{E}X = \Omega(n^v p^e) = \omega(1)$ and

$$\text{Var}(X) \leq \mathbb{E}X + \sum_S \Pr(E_S) \sum_{T \sim S} \Pr(E_T|E_S)$$

for any fixed $S$, we want $\sum_{T \sim S} \Pr(E_T|E_S) = o(\mathbb{E}X)$.

Note that $E_S$ and $E_T$ are dependent ($S \sim T$) iff $|S \cap T| \geq 2$. We have

$$\sum_{T \sim S} \Pr(E_T|E_S) = \sum_{i=2}^{v-1} \sum_{|T \cap S| = i} \Pr(E_T|E_S)$$

$$= \sum_{i=2}^{v-1} O(n^{v-i})O(p^{e-iv/v})$$

$$= \sum_{i=2}^{v-1} O((n^v p^e)^{1-i/v})$$

$$= O(n^v p^e) = o(\mathbb{E}X)$$

as desired. (We break up the summation according to $i$ the number of points in $T \cap S$. Then, given $S$, there are $O(n^{v-i})$ ways to choose $T$. There are $O(1)$ possible copies of $H$ in $T$ and each has at most $ie/v$ edges in $T \cap S$ since $H$ is balanced. Thus $H$ has at least $e-iv/v$ edges outside of $T \cap S$ and hence $\Pr(E_T|E_S) = O(p^{e-iv/v})$.)

2. [4] We begin by noting that we may assume that the out-degree of each vertex is exactly $\delta$ (to see this, consider the deletion of arcs).

**Claim.** There exists $f : V \to \{0, \ldots, k - 1\}$ such that for all $v \in V$ there exists $w \in V$ such that $(v, w) \in A$ and $f(w) = f(v) + 1 \mod k$. 

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It follows from the existence of this map $f$ that there exists a sequence of distinct vertices $v_1, \ldots, v_\ell$ such that

$$(v_i, v_{i+1}) \in A \quad \text{and} \quad f(v_{i+1}) = f(v_i) + 1 \mod k \quad \text{for } i = 1, \ldots, \ell - 1$$

and $(v_\ell, v_1) \in A$ and $f(v_1) = f(\ell) + 1 \mod k$. Note that this gives a directed cycle and $\ell$ is a multiple of $k$.

It remains to prove the claim.

Proof. We use the general Lovász Local Lemma given in lecture: if $A_1, \ldots, A_m$ are events such that

- there is a digraph $D$ on $[m]$ with $A_i$ mutually independent of all the events $A_j$ with $i \not\rightarrow j$
- there are $x_i \in [0,1)$ with $Pr(A_i) \leq x_i \prod_{j \leftarrow i} (1 - x_j)$ for all $i$,

then

$$Pr\left(\bigcap_{i=1}^m A_i\right) \geq \prod (1 - x_i) > 0.$$ 

Note if the maximum outdegree in $D$ is $d \geq 2$ and each $Pr(A_i) \leq p$ and $4pd \leq 1$ then we can take each $x_i = 1/d$ and the conclusion will hold.

Now, for $v \in V$ define

$$N^+(v) = \{w \in V : (v, w) \in A\}.$$ 

Let $f : V \rightarrow \{0, \ldots, k - 1\}$ be a map chosen uniformly at random (for each $v \in V$, $f(v)$ is chosen uniformly and independently at random from $\{0, \ldots, k - 1\}$). For $v \in V$ let $A_v$ be the event that there does not exist $w \in N^+(v)$ such that $f(w) = f(v) + 1$ modulo $k$. Note that

$$Pr(A_v) = \left(1 - \frac{1}{k}\right)^\delta =: p.$$ 

For $v, u \in V$ set $v \rightarrow u$ in $D$ iff $u \in N^+(v)$ or $N^+(v) \cap N^+(u) \neq \emptyset$. Note that conditioning on the events $A_u$ with $v \not\rightarrow u$ has no bearing on the value of $f$ on $N^+(v)$ (indeed, setting $f(w)$ for all $w \notin N^+(v)$ does not influence the probability of $A_v$). Note that $d$ the maximum degree in the directed graph $D$ so created is at most $\delta \Delta$ (there are $\delta$
out-neighbors of $v$ and for a given out-neighbor $u$ of $v$ there are at most $\Delta - 1$ other vertices that also have $u$ as an out-neighbor). The Claim then follows from the version of the local lemma stated above. \hfill \Box

3. [2] We have $R(4, k) \geq n - \binom{n}{4}p^6 - \left(\binom{n}{k}\right)^{\binom{k}{2}}$. Wanting both $n$ and $\binom{n}{4}p^6$ to be $\Theta((k/\log(k))^2)$ it becomes clear that one should use $p = \log(k)/k$. Then $a = \binom{n}{4}p^6 - \left(\binom{n}{k}\right)^{\binom{k}{2}} \leq n^4/4!p^6 + (ne/k)^k p^k$ for $k \geq 3$ (since we have $\binom{k}{2} \geq k$). Plugging in $n = k^2/\log^2(k)$ and $p = k/\log(k)$ gives $a \leq k^3/(4!\log^2(k)) + (e/\log(k))^k$ so $R(4, k) \geq (23/24)(k^2/\log^2(k))$.

4. [3]

(a) If we give vertex $x_i, y_i$ the color list $[3] - i$, the graph is not list colorable. The total number of colors used by the vertices of one part has to be two or more since for each color there is a vertex which can’t use it. But if the total number used is two or more, two of those colors will form the color list of a vertex in the other part which then can’t be colored.

On the other hand if there are 3 colors in each list we can make a list coloring. If there is a repeated color in the lists of one side, color that side using that color and possibly another. Since at most two colors have been used, coloring the vertices on the other side is a trivial matter. On the other hand, if all the colors on one side are distinct, there are 27 ways to color the vertices. We can choose one of these colorings so that it does not contain any color list from the other side (there are only 3 lists to avoid). Then coloring the other side is easy.

(b) The example of the first part can be generalized to $\text{ch}(K_{m,m}) > k + 1$ where $m = \binom{2k+1}{k+1}$. Give the vertices $x_i, y_i$ the color list that is the $i$th $k + 1$ subset of $[2k + 1]$. The resulting graph can’t be list-colored. If it could the total number of colors used on one side must be $k + 1$ or more because every subset of $k$ colors is missing from some color list on that side. But then there is a vertex on the other side having $k + 1$ of those colors as its color list, which then can’t be colored.)

Note $n_k := 2\binom{2k+1}{k+1} \geq 2^{2k+2}$ so we have just given a graph $G_{n_k}$ with $\text{ch}(G_{n_k}) > k + 1 \geq (1/2)\log_2(n_k)$. To extend this to a a graph with $n_{k+1} > n > n_k$ just pad $G_{n_k}$ with $n - n_k$ isolated vertices.
5. Assume all lists are of size \( k = \log_2(n) + 1 \). Let \( C = \bigcup_{v \in V} S(v) \) be the set of colors. Let \( C_X, C_Y \) be a random partition of \( C \) into two parts (possibly empty), i.e. for each \( c \in C \) let \( \Pr(c \in C_X) = 1/2 \) independently and \( C_Y = C - C_X \). The idea here is that \( C_X \) will be colors used on vertices in \( X \) and \( C_Y \) colors used on \( Y \). For each \( x \in X \) \((y \in Y)\) let \( E_x \) \((E_y)\) be the event that \( S(x) \cap C_X = \emptyset \) \((S(y) \cap C_Y = \emptyset)\). Let \( I = \sum_{x \in X} I_x + \sum_{y \in Y} I_y \). Where \( I_x \) is the indicator of \( E_x \), etc. We have \( \mathbb{E}I = n\mathbb{E}I_x = n(1/2)^k < n(1/n) = 1 \) and hence there exists a list-coloring.

6. For each \( v \in V \), delete colors from each \( S(v) \) until \( |S(v)| = 10d \). For each \( v \in V \), independently pick \( f(v) \) uniformly at random from \( S(v) \). Let \( E_0 \) be the set of edges \( \{v, w\} \in E \), with \( S(v) \cap S(w) \neq \emptyset \). For each \( e = \{v, w\} \in E_0 \), \( c \in S(v) \cap S(w) \) let \( A_{e,c} \) be the event that \( f(v) = f(w) = c \). We apply the Lovasz Local Lemma to the \( A_{e,c}'s \) to show that there is a coloring that avoid all of them.

Clearly \( \Pr(A_{e,c}) = (1/10d)^2 =: p \). Set \( A_{e,c} \sim A_{e',c} \) iff \( e \) and \( e' \) are incident. Since setting the color of \( f \) on every vertex outside of \( e \) does not affect \( A_{e,c} \) this definition of \( \sim \) gives a dependency graph on the events \( A_{e,c} \). The maximum degree of a vertex \( A_{e,c} \) in this graph can be calculated as follows. Each vertex \( v \) and \( w \) of \( e \) has a list of \( 10d \) colors and for each color, \( d \) neighbors that share that color. Hence the number of edges events \( A_{e',c} \) with \( e' \) incident to \( e \) is \( \leq (2)(10d)(d) = 20d^2 \). We see that \( ep(d + 1) = e(20d^2 + 1)/100d^2 < 1 \) and hence there is a good coloring.