1. (a) Fix $l$. We prove the statement by induction on $k$. The case $k = 1$ is trivial. A sequence of length $l + 1$ is either decreasing or contains an increasing subsequence of length 2. Now we assume that the theorem has been proved for all $1 \leq k < K$. Assume that we have a sequence $s$ of length $Kl + 1$ with no decreasing subsequence of length $l + 1$. Consider all the members of the sequence $s$ that occur as an endpoint of an increasing subsequence of maximum length. These endpoints must form a decreasing subsequence and so, by assumption, there are at most $l$ of them. Removing these endpoints from $s$ gives a sequence $s'$ of $\geq (K - 1)l + 1$ elements. By assumption, $s'$ does not contain a decreasing subsequence of length $l + 1$, so by induction, it must contain an increasing subsequence $i$ of length $\geq K$. Since $i$ does not involve any endpoints, it is not an increasing sequence of maximum length in $s$. Thus $i$ extends to an increasing sequence of length $\geq K + 1$ in $s$.

Another way. For each $1 \leq n \leq kl + 1$ let $a_n$ be the $n$th element of the sequence and let $i_n(d_n)$ be the length of the longest possible increasing(decreasing) subsequence ending in $a_n$. If the theorem is false then we have an example of a sequence, where $1 \leq i_n \leq k$ and $1 \leq d_n \leq l$ for all $1 \leq n \leq kl + 1$. By the pigeonhole principle, this means there exist $m < n$ such that $(i_m, d_m) = (i_n, d_n)$. This is a contradiction though. If $a_m < a_n$, $i_m < i_n$ because there is an increasing subsequence of length $i_m$ ending in $a_m$ which when followed by $a_n$ becomes an increasing subsequence of length $i_m + 1$ ending in $a_n$. Similarly if $a_m > a_n$ then $d_m < d_n$. (Now you could show that a $klm + 1$-sequence has a $k + 1$-increasing, $l + 1$-decreasing, or a $m + 1$-constant subsequence.)

(b) $s = (l, l - 1, l - 2, \ldots, 1, 2l, 2l - 1, \ldots, l + 1, \ldots, kl, kl - 1, \ldots, kl - l + 1)$ but there are many possibilities. Can you enumerate them? I am not sure if this is an doable problem.

2. (a) If $n \geq m^2 + 1$ then you can divide the square into an $m \times m$ grid of smaller squares. One of these smaller $(1/m) \times (1/m)$ squares will receive at least two shots. The largest distance that could possibly separate these two of these shots is the diameter of the
square or $\sqrt{2}/m$. Since this analysis works if $m \leq \sqrt{n-1}$ we have an upper bound of $\sqrt{2}/\lfloor \sqrt{n-1} \rfloor$ as claimed.

(b) Same thing, but here we need $n \geq 2m^2 + 1$ in order to ensure that a small square gets at least three shots. The area $A$ of the triangle that these three points form is smaller than the area of the largest possible triangle contained in an $(1/m) \times (1/m)$ square, or $1/(2m)$. Thus if $m \leq \lfloor \sqrt{(n-1)/2} \rfloor$, we have $A < 1/(2m)$ which gives the claimed upper bound.

3. Let the edges of $K_ω$ be colored red or blue. We iteratively construct a subgraph $\{w_1, w_2, \ldots \}$ having the property that the color of the edge $\{w_i, w_j\}$ with $i < j$ depending only on $w_i$. First let $\{v_1, v_2, \ldots,\} = \{1, 2, \ldots,\}$. There must be an infinite subsequence of $\{v_2, \ldots,\}$ connected to $v_1$ by red edges.

Fix $v_1 = 1$. If $v_1, \ldots, v_i$ have already been fixed, we show how to produce $v_{i+1}$. Let $R_i (B_i)$ be the set of vertices (not including $v_1, \ldots, v_{i-1}$) that are adjacent to $v_i$ through a red (blue) edge. Note, that either the set $R_i$ or $B_i$ is infinite. If $R_i$ is infinite throw away $B_i$, otherwise throw away $R_i$. In either case, infinitely many vertices remain besides $v_1, \ldots, v_i$ and $v_i$ is connected to all of them through edges of the same color. Let $v_{i+1}$ be the first vertex that remains, and continue.

4. Given a red-blue coloring $c$ on $E(K_{n,n})$, where $K_{n,n}$ has bipartition $X \cup Y$, $X = \{x_1, \ldots, x_n\}$, $Y = \{y_1, \ldots, y_n\}$, place the following coloring on $E(K_n)$: $c(\{i, j\}) = c(\{x_i, y_j\}), i < j$. If $n \geq R(2k, 2k)$ we have a monochromatic clique of size $2k$ in $K_n$, on vertices $i_1 < \cdots < i_k < j_1 < \cdots < j_k$. But this means $\{x_{i_m}, y_{j_n}\}, 1 \leq m, n \leq k$ are all monochromatic. So we have a monochromatic $K_{k,k}$ if $n \geq R(2k, 2k)$.

Another approach, one that avoids $R(k, k)$. Pick $x_1$, notice $x_1$ will be adjacent to at least half of the vertices in $Y$ through red edges or at least half of the vertices in $Y$ through blue edges. Restrict attention to this “monochromatic” subset of $Y$ of size $\geq n/2$. Similarly $x_2$ will be adjacent to $\geq 1/2$ of these points monochromatically, restrict attention to these $\geq n/4$ points and continue. After $2k - 1$ rounds, we will have $x_1, \ldots, x_{2k-1}$ adjacent to a subset $Y'$ of $n/2^{2k-1}$ points in $Y$ such that the color of an edge depends only on its endpoint $x_i$. Since there are
2k − 1 of these points $x_i$ in all, at least $k$ of them must determine the same color. So as long as $n/2^{2k−1} \geq k$ we have a monochromatic $K_{k,k}$.

5. [3] Let $T$ be an tree having $a$ vertices (a tree is a connected graph containing no cycles).

(a) Fix $a \geq 2$. We prove the statement by induction on $b \geq 2$. If $b = 2$ then we have $n \geq a$ vertices, and so either there is a blue edge ($K_2$) or a red copy of $K_a$ and thus of $T$. Suppose now the statement has been proved for $b = B−1$. We now prove it for $b = B$. Suppose, for sake of contradiction, that we have $n = (a−1)(B−1) + 1$ vertices but no red $T$ and no blue $K_B$. Suppose a vertex $v$ has blue degree (number of blue edges adjacent to the $v$) $\geq (a−1)(B−2) + 1$. By induction, the endpoints of these edges contain a red $T$ or a blue $K_{B−1}$. By assumption, neither outcome is allowed (a blue $K_{B−1}$ together with $v$ forms a blue $K_B$.) Thus every vertex has blue degree $\leq (a−1)(B−2)$ or, what is the same, red degree $\geq a−1$. We claim that this means there is a red $T$ which is a contradiction, completing the proof.

In fact we claim that a graph in which every degree is $\geq a−1$ contains every possible tree $T$ on $a$ vertices. Clearly this is true for $a = 2$. Suppose the claim has been proven for $a = A−1$, we now prove it for $a = A$. Let $G$ be a graph with $\delta \geq A−1$. Let $T$ be a tree on $A$ vertices. Pick a leaf $v$ of $T$ and remove it to get a tree $T'$ on $A−1$ vertices. Remove a vertex $x$ from $G$. The resulting graph $G'$ has $\delta \geq A−2$. Thus $G'$ contains a copy of $T'$. Let $w$ be the vertex in $T'$ adjacent to $v$. Notice that $\deg_{T'}(w) \leq A−2$. Thus either $w$ is adjacent to $x$, in which case $T'$ together with $x$ form a copy of $T$ in $G$, or $w$ is not adjacent to $x$, in which case $\deg_{G'}(w) \geq A−1$, and so there is a neighbor $x'$ of $w$ in $G'$ outside of $T'$, in which case $T'$ together with $x'$ is a copy of $T$.

(b) Consider a union of $b−1$ disjoint copies of red $K_{a−1}$'s. Color the remaining edges blue.

6. For positive integers $k, r$ let $W(k, r)$ be the least $N$ such that any $r$-coloring of $[N]$ contains a monochromatic $k$-term arithmetic progression (these are the so-called Vander Waerden numbers, the existence of which is given by Vander Waerden’s Theorem).
We claim that

$$\{(x, y) \in \mathbb{N}^2 : 1 \leq x \leq W(k, r), 1 \leq y \leq W(k, r^{W(k,r)})\}$$

contains a two-dimensional arithmetic progression of order $k$. This follows from two applications of the definition of the Vander Waerden numbers. First, we define a coloring

$$g : \{0 \leq a \leq W(k, W(k, r^{W(k,r)}))\} \rightarrow [r]^{W(k,r)},$$

(recall that $[r]^{W(k,r)}$ is the set of all functions from $[W(k, r)]$ to $[r]$) by setting $g(a)(x) = f(x, a)$. In other words, we ‘color’ the row $y = a$ with the coloring that row $y = a$ gets under $f$. It then follows that there exists $a_2$ and $d_2$ such that

$$g(a_2) = g(a_2 + d_2) = \cdots = g(a_2 + (k-1)d_2)$$

from which it follows that

$$f(x, a_2) = f(x, a_2 + d_2) = \cdots = f(x, a_2 + (k-1)d_2) \quad (1)$$

for $1 \leq x \leq W(k, r)$. Now, we define a coloring $h : 1 \leq a \leq W(k, r) \rightarrow [r]$ by setting $h(a) = f(a, a_2)$. For this coloring there exists $a_1$ and $d_1$ such that

$$h(a_1) = h(a_1 + d_1) = \cdots = h(a_1 + (k-1)d_1)$$

which implies

$$f(a_1, a_2) = f(a_1 + d_1, a_2) = \cdots = f(a_1 + (k-1)d_1, d_2). \quad (2)$$

It follows from (1) and (2) that $(a_1, a_2), d_1$ and $d_2$ define a monochromatic two-dimensional arithmetic progression of order $k$ in $[1, W(k, r)] \times [1, r^{W(k,r)}]$ with respect to the original coloring $f$.

7. Let $n \geq HJ(r, t^m)$ and set $N = mn$. Let

$$\varphi : [t]^m \rightarrow [t^m]$$

be an arbitrary fixed bijection. We identify $[t]^N$ and $[t^m]^n$ by identifying the vector

$$(x_1, \ldots, x_m, x_{m+1}, \ldots, x_{2m}, \ldots, x_{(n-1)m+1}, \ldots, x_{nm})$$
in \([t]^N\) with the vector
\[
(\varphi(x_1, \ldots, x_m), \varphi(x_{m+1}, \ldots, x_{2m}), \ldots; \varphi(x_{(n-1)m+1}, \ldots x_{nm}))
\]
in \([t^m]^n\). Given a coloring
\[
\sigma: [t]^N \rightarrow [r],
\]
this identification gives a coloring
\[
\sigma': [t^m]^n \rightarrow [r]
\]
in the natural way. It follows from the Hales Jewett-Theorem that \(\sigma'\) has a monochromatic line. In other words, there exists a partition \(X \cup Y = [n]\) and an element \(v_x \in t^m\) for each \(x \in X\) such that \(\sigma'\) is constant on the set of vectors that agree with \((v_x : x \in X)\) on \(X\) and are constant on \(Y\).

For \(i = 1, \ldots, m\) let
\[
I_i = \{m(x - 1) + i : x \in X\}
\]
and
\[
I_0 = \bigcup_{y \in Y} \{m(y - 1) + j : j = 1, \ldots, m\}.
\]
These sets of coordinates define a monochromatic \(m\)-space in \([t]^N\).