1. [3] Let $f(d)$ be the maximum number $k$ ever necessary to split a set $S \subseteq \mathbb{R}^d$ with $\text{diam}(S) = 1$ into $k$ pieces $S = \bigcup_{i=1}^{k} S_i$ where each $S_i$ has $\text{diam}(S_i) < 1$. Taking $S$ to be the $d + 1$ vertices of a regular simplex (think equilateral triangle when $d = 2$), it is easy to see $f(d) \geq d + 1$. It is known that $f(d) \leq 2^d$. In 1933 Borsuk conjectured $f(d) = d + 1$. Try proving $f(2) = 3$ or $f(3) = 4$. The conjecture was widely believed and proven in several low dimensions and in high dimensions for centrally symmetric $S$ and other restricted cases.

Then Kahn and Kalai showed in 1992 that $f(d) \geq (1.2)^{\sqrt{d}}$ for $d$ large enough. Here’s an important part of how they did it. Let $X = \binom{[4p-1]}{2p-1}$ where $p$ is a prime number. For each $A \in X$ let $E_A$ be the set of edges from $A$ to $\overline{A}$ in $\binom{[4p-1]}{2p-1}$. Let $s_A \in \{0, 1\}^d$ be the indicator vector of $E_A$, where $d = \binom{4p-1}{2}$. Let $S = \{ s_A : A \in X \}$.

(a) Show that the distance between $s_A$ and $s_B$ is maximized when $|A \cap B| = p - 1$.
(b) What is the maximum possible size of $S' \subset S$ with $\text{diam}(S') < \text{diam}(S)$? (Pull it back to a question about $X$.)
(c) So this example shows that $f(d)$ is at least how big? For which $d$ does this first show $f(d) > d + 1$?

2. [4] Let $2 \leq k < n$, and $T$ be a tree on $k$ edges. The Erdős-Sós conjecture is that $\text{ex}(n, T) \leq (k - 1)n/2$

(a) Show that $\text{ex}(n, T) \leq (k - 1)n$. (Hint: one way is to start by showing that $d(G - v) \geq d(G)$ iff $d(v) \leq d(G)/2$ and use this to build a high degree subgraph of $G$.)
(b) Prove the conjecture when $T$ is a star.
(c) Prove the conjecture when $T$ is a path. (First prove that if $G$ is connected then $G$ contains a path of length $\min(2\delta(G), |G| - 1)$.)

(a) Let $r < n$ be positive integers. Let $f \in \mathbb{R}[x_1, \ldots, x_n]$ be a polynomial with the property that $f(1_A) \neq 0$ for all $A \subseteq [n]$ such that $|A| \leq r$. For $I \subseteq [n]$ let

$$x_I = \prod_{i \in I} x_i.$$  

(So, $X_\emptyset = 1$.) Prove that the set of polynomials

$$\{fx_I : I \subseteq [n], |I| \leq r\}$$

is linearly independent in $\mathbb{R}[x_1, \ldots, x_n]$.

(b) Use (a) to prove the Ray-Chaudhuri – Wilson Theorem: If

i. $L$ is a set of $s$ integers,

ii. $\mathcal{F} \subseteq \binom{[n]}{k}$, and

iii. $A, B \in \mathcal{F}, A \neq B \Rightarrow |A \cap B| \in L$

then

$$|\mathcal{F}| \leq \binom{n}{s}.$$  

Hint. Introduce ‘extra’ polynomials that do not spoil the linear independence of the collection.

4. [4] Prove Turán’s Theorem (i.e. show that

$$\max\{|E(G)| : V(G) = [n] \text{ and } K_{s+1} \not\subseteq G = |E(T_{n,s})|$$

where $T_{n,s}$ is the corresponding Turán graph) in the following two ways.

(a) Assume $s$ divides $n$ and consider the (quadratic programming) problem

$$\max \sum \{\lambda(x)\lambda(y) : x, y \in V(G), x \sim y\}$$

subject to

$$\lambda(x) \geq 0 \quad \forall x \in V(G) \quad \sum \lambda(x) = 1.$$
(b) Show that

$$\alpha(G) \geq \sum_{x \in V(G)} \frac{1}{d(x) + 1}$$

Hint: For each of the n! orderings of V form an independent set using the greedy algorithm.

5. [2] Show that for every k there exists a tournament $T = (V, A)$ such that for all $X \subset V$ of cardinality k and all $Y \subset X$ there exists $v \in V$ such that

$$(x \in X \setminus Y \Rightarrow (x, v) \in A) \text{ and } (x \in Y \Rightarrow (v, x) \in A).$$

In words, v beats every element of $Y$ and every element of $X \setminus Y$ beats v.

6. [3] Let the color Ramsey number of a graph $H$, $R^*(n, H)$, be the minimum number $k$ of colors necessary to color the edges of $K_n$ in order to avoid a monochromatic copy of $H$. Suppose $G$ is a graph on $t$ edges that contains no copy of $H$. Show that $R^*(n, H) \leq n^2 \log(n)/t$ for $n$ large enough.

7. [3] Let $G$ be a graph on $[n]$ with vertex degrees $d_1, \ldots, d_n$ and average degree $d = (1/n) \sum d_i$.

(a) Show that $\tau(G) \leq n - \sqrt{2n/d}$. Pick a set $C$ uniformly at random from $\binom{[n]}{n'}$. Pick $n'$ to force $\mathbb{E}X < 1$ where $X$ is the number of edges left uncovered by $C$.

(b) Pick $C$ with $\Pr(i \in C) = p$ independently for each $i$. Pick $p$ to force $\mathbb{E}X < 1$. This seems to make the calculations easier, can you show $\tau(G) \leq n - \sqrt{2n/d}$, this way? (If you are working on this part and are getting stuck, make sure you ask me about it!)

(c) Pick $C$ as in (b). But now only view it as a partial cover. Form a cover by performing some alterations on $C$. Prove that $\tau(G) \leq n(1 - 1/(2d))$.

(d) A good cover should favor higher degree vertices over lower degree vertices. If $\sigma \in S_n$ show $C = C(\sigma) = \{i : \exists j \sim i \ \sigma(i) < \sigma(j)\}$ is a vertex cover. If $\sigma$ is chosen uniformly at random, what is $\Pr(i \in C)$ in terms of $d_i$? Show that $\tau(G) \leq n(1 - 1/(d+1))$. Show this is tight.