1. [3] (Magic permutation proofs.)

(a) Show that if $\mathcal{F} \subset 2^{[n]}$ is an antichain then $\sum_{A \in \mathcal{F}} \binom{n}{|A|}^{-1} \leq 1$. Show that this implies Sperner’s theorem. (Hint: For each $A \in \mathcal{F}$ count the permutations whose first $|A|$ members are the elements of $A$.)

(b) Let $\mathcal{F} = \{(A_1, B_1), \ldots, (A_k, B_k)\}$ be a family of pairs of subsets of $[n]$ such that $|A_i| = a$ and $|B_i| = b$ for all $1 \leq i \leq k$ and $A_i \cap B_j = \emptyset$ if and only if $i = j$. Prove that $|\mathcal{F}| \leq \left(\begin{array}{c} a + b \\ b \end{array}\right)$. Notice that this upper bound is independent of $n$. You should produce an example showing that it is tight. (Hint: for each $i$ count the permutations that have every element of $A_i$ before every element of $B_i$.)

2. [3] (An inclusion-exclusion statement.)

(a) Prove that for $n, k \geq 0$,

$$a_{n,k} := \sum_l \binom{n}{l} \binom{l}{k} (-1)^{l-k} = \begin{cases} 1, & n = k \\ 0, & - \end{cases}$$

(b) Given the inclusion exclusion setup $(A_1, \ldots, A_n \subset A)$ for $x \in A$, let $\nu(x) = |\{i : x \in A_i\}|$. Prove the following generalization of the inclusion exclusion statement,

$$|\{x \in A : \nu(x) = k\}| = \sum_{I \subseteq [n]} (-1)^{|I|-k} \binom{|I|}{k} |A_I|.$$ 

(c) For $n, r \geq k \geq 0$, let $f_{n,r,k}$ be the number of functions $f : [n] \rightarrow [r]$ with range of size $k$. Using (b) find a formula for $f_{n,r,k}$. Using that formula, show that $f_{n,r,k} = f_{n,k,k}(r)$. Why should this be true?


The game is played on a hypergraph $H = (V, E)$ set of points $V$. Two players alternate claiming points of $V$, player one going first. Once claimed, a point cannot be claimed by another player. The first player
to claim all the points of some $A \in E$ wins the game. Notice, that
in the following example of $[5]^2$ tic-tac-toe (the winning sets are the
horizontal, vertical and diagonal lines), the second player to move has
a “draw force pairing strategy” to prevent the first player from winning.
Namely, at each round, if player one takes a square marked $1 \leq i \leq 12$,
then player two takes the other square marked $i$, if it is still available,
and otherwise chooses some other unclaimed point.

\[
\begin{array}{cccccc}
1 & 6 & 10 & 7 & 7 \\
11 & 3 & 3 & 2 & 12 \\
9 & 6 & * & 5 & 9 \\
11 & 2 & 4 & 4 & 12 \\
8 & 8 & 10 & 5 & 1 \\
\end{array}
\]

Suppose $H = (V, E)$ is $k$-uniform (every $A \in E$ has $|A| = k$) and $d$-
regular (every $x \in V$ is contained in exactly $d$ members of $E$). Show
that if $k \geq 2d$ then the second player has a draw force pairing strategy.

4. [3] Let $G$ be a graph on $n$ vertices. If $S \subseteq V(G)$, we define
the deficiency of $S$ to be $\delta(S) = q(G - S) - |S|$, where $q(H)$ is the number
of odd components of $H$. We define the defect of $G$, to be $d(G) := \max_{S \subseteq V(G)} \delta(S)$. Prove that $2\nu(G) = n - d(G)$. In other words, a
maximum size matching covers all but $d(G)$ of the points of $G$. (Hint: mimic the defect proof given of the Konig-Egervary theorem, add a
clique $K_d$ of $d = d(G)$ points to $G$ and connect every point in $G$ to
every point in $K_d$.)

5. [4] For $k \in \mathbb{N}$, $r \in \mathbb{P}$ and $R$ a subset of $[k + r]$ of cardinality $r$, let
$f(k, r)$ be the number of rooted-forests on vertex set $[k + r]$ for which
$R$ is the set of roots (i.e. $f(k, r)$ is the number of forests $F$ such that
each connected component of $F$ contains exactly one element of $R$).

(a) Prove the recurrence

\[
f(k, r) = \sum_{s=1}^{k} \binom{k}{s} r^s f(k - s, s).
\]

(b) Use (a) to prove Cayley’s Formula.
6. [3] Let \( r, c \leq n \) be positive integers. A *Latin rectangle* is an \( r \times c \) matrix with entries from \([n]\) such that each \( i \in n \) appears at most once in each row and column. Such a rectangle with \( r = c = n \) is a *Latin square*. Show that an \( r \times c \) Latin rectangle extends (in the obvious sense) to a Latin square iff it uses each symbol at least \( r + c - n \) times.