1. [4] A threshold graph is a simple graph which may be defined inductively as follows, (i) an empty graph is a threshold graph, (ii) the disjoint union of a single vertex with a threshold graph is a threshold graph, (iii) the complement of a threshold graph is a threshold graph. Let \( t_n \) be the number of threshold graphs on \([n]\), i.e. \( \{t_n\}_{n=0}^\infty = \{1,1,2,8,46,\ldots\} \), and let \( s_n \) be the number of threshold graphs on \([n]\) with no isolated vertex, i.e. \( \{s_n\}_{n=0}^\infty = \{1,0,1,4,23,\ldots\} \).

   (a) Let \( t(x), s(x) \) be the exponential generating functions for \( t_n, s_n \) respectively. Show that \( t(x) = e^x s(x) \) and \( t(x) = 2s(x) + x - 1 \) so that \( t(x) = e^x (1 - x)/(2 - e^x) \) and \( s(x) = (1 - x)/(2 - e^x) \).

   (b) Let \( c_n \) be the number of ordered partitions of \([n]\) (i.e. the partition \( 12\mid 34 \) of \([4]\) gives rise to two different ordered partitions \( (12\mid 34) \) and \( (34\mid 12) \)). Show that the exponential generating function of \( c_n \) is given by \( c(x) = 1/(2 - e^x) \), by proving

   \[
   c_n = \sum_{i=1}^{n} \binom{n}{i} c_{n-i} \text{ for } n \geq 1, \quad c_0 = 1.
   \]

   (c) Given (a) this implies \( s_n = c_n - nc_{n-1} \) for \( n \geq 1 \). Give a direct combinatorial proof.

2. [3] Let \( k > 2 \). Give a generating function proof that the number \( f_k(n) \) of permutations \( \pi \in S_n \) all of whose cycle lengths are divisible by \( k \) is given by

   \[
   f_k(n) = 1 \cdot 2 \cdot 3 \cdots (k - 1)(k + 1)^2(k + 2) \cdots (2k - 1)(2k + 1)^2(2k + 2) \cdots (n - 1) \text{ if } k | n \text{ and is 0 otherwise.}
   \]

   (b) Give a combinatorial proof of (a).

3. [3] Let \( \Delta \) be a set \( \{\delta_1, \ldots, \delta_n\} \) of \( n \) straight lines in the planes lying in general position (no two parallel, no three intersecting in a point). Let \( P \) be the set of their points \( \delta_i \cap \delta_j \) of intersection, so \( |P| = \binom{n}{2} \). A cloud is an \( n \) element subset of \( P \) containing no three collinear points. Let \( c_n \) be the number of possible clouds. Find the exponential generating function \( c(x) \) of \( c_n \), as follows:

   (a) Find a bijection between the set of clouds and the set of regular simple graphs on \([n]\) of degree 2.
(b) Find the exponential generating function for the number $g_n$ of 2-regular simple graphs on $[n]$.

4. [3] Combinatorial trigonometry? A permutation $a_1a_2 \cdots a_n$ of $[n]$ is alternating if $a_1 < a_2 > a_3 < a_4 > \cdots$. Let $b_n$ be the number of alternating permutations if $n$ is odd, $e_n$ the number if $n$ is even ($e_0 = 1$). Let $b(x)$ and $e(x)$ be the exponential generating functions of $b_n$ and $e_n$ respectively. (The numbers $b_n$ and $e_n$ are called Bernoulli and Euler numbers respectively.)

(a) Show that if $n \geq 1$, $b_{n+1} = \sum_k \text{odd} \binom{n}{k} b_k b_{n-k}$ if $n + 1$ is odd, and $e_{n+1} = \sum_k \text{even} \binom{n}{k} e_k b_{n-k}$ if $n + 1$ is even. (Consider the location of $n + 1$ in the permutation.) Deduce that $b(x) = \tan(x)$, $e(x) = \sec(x)$.

(b) Give a combinatorial proof of $1 + \tan^2(x) = \sec^2(x)$. (Give a bijection between the structures counted by the e.g.f $1 + b^2(x)$ and the structures counted by the e.g.f $e^2(x)$.)

5. [4]

(a) If $\sum a_n x^n / n! = \exp(x + x^2/2)$, find a simple closed form for

$$\sum_{i=0}^{n} (-1)^i \binom{n}{i} a_i.$$  

(Do this via generating functions. Don’t use what we know about $a_n$.)

(b) Use the exponential formula to show that $a_n$ as in (a) is the number of involutions in the symmetric group $S_n$.

(c) Use (b) to give a combinatorial proof of your formula from (a).

6. [2] For a random variable $\xi$ taking values in $\mathbb{N}$, we write $P(x) = P_\xi(x)$ for the probability generating function for $\xi$:

$$P(x) = \sum_{n \geq 0} Pr(\xi = n) x^n.$$  

(a) Express $E[\xi]$ in terms of $P(x)$. 

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(b) Let $\xi_1, \ldots, \xi_t$ be independent random variables and $\xi = \sum_{i=1}^{t} \xi_i$. Express $P_{\xi}(x)$ in terms of the $P_{\xi_i}(x)$'s.

(c) Suppose $\xi$ takes values in $\{0,1,\ldots,n\}$ (so $P$ is a polynomial). Show that if the roots of $P$ are real, then $\xi = \sum_{i=1}^{n} \xi_i$ where $\xi_1, \ldots, \xi_n$ are independent (but not necessarily identical) 0-1 random variables.

7. [3] Let $t_n$ denote the number of non-congruent triangles whose sides have integer length and whose perimeter is $n$. For instance, $t_9 = 3$, corresponding to $3 + 3 + 3, 2 + 3 + 4, 1 + 4 + 4$. Find $\sum_{n \geq 3} t_n x^n$. 
