1. (5.3.4) Prove that the chromatic polynomial of $C_n$ is

$$\chi(C_n; k) = (k - 1)^n + (-1)^n(k - 1).$$

We prove the statement by induction on $n$. If $n = 3$, $\chi(C_n; k) = \chi(K_3; k) = k(k - 1)(k - 2) = k^3 - 3k^2 + 2k = (k - 1)^3 + (-1)^3(k - 1)$.

Suppose now that $n > 3$ and that the statement has been proven for $3 \leq k < n$. Let $e$ be any edge of $C_n$. We know that $\chi(C_n; k) = \chi(C_{n-e}; k) - \chi(C_n \cdot e; k)$. We know that $\chi(C_{n-e}; k) = \chi(P_n; k) = k(k-1)^{n-1}$. By induction, we also know that $\chi(C_n \cdot e; k) = \chi(C_{n-1}; k) = (k-1)^{n-1} + (-1)^{n-1}(k-1)$. Putting these two facts together we get $\chi(C_n; k) = \chi(C_{n-e}; k) - \chi(C_n \cdot e; k) = k(k-1)^{n-1} - ((k-1)^{n-1} + (-1)^{n-1}(k-1)) = (k-1)^n + (-1)^n(k-1)$.

2. (5.1.7 and 5.1.35) We know greedy colorings of $G$ always produce colorings using $k \leq \Delta(G) + 1$ colors. Here we’ll show that $\Delta(G) + 1$ is sometimes obtained. Let $G = P_4$. Demonstrate how a particular ordering of the vertices results in a greedy coloring using 3 colors. Give an example of a bipartite $G$ and a greedy coloring of it using 4 colors.

For the first part, imagine $G = P_4$ consists of $V(G) = \{a, b, c, d\}$, $E(G) = \{ab, bc, cd\}$. Suppose the vertices are ordered from first to last as d, a, b, c. The resulting greedy coloring for the vertices is $f(d) = 1$, $f(a) = 1$, $f(b) = 2$, $f(c) = 3$ (or $f(a) = 1$, $f(b) = 2$, $f(c) = 3$, $f(d) = 1$).

For the second graph, there are many possible examples. But here is one consisting of two $P_4$’s linked by a single edge. $V(G) = \{a, b, c, d, A, B, C, D\}$, $E(G) = \{ab, bc, cd, AB, BC, CD, cC\}$. Order the vertices d, a, b, c, D, A, B, C. The resulting greedy coloring will be $f(d) = 1$, $f(a) = 1$, $f(b) = 2$, $f(c) = 3$, $f(D) = 1$, $f(A) = 1$, $f(B) = 2$, $f(C) = 4$.

3. Suppose we’d like a simple graph $G$ on $n = |V(G)|$ vertices to have independence number $\alpha(G) \leq 2$. We could have $G$ be the complete graph, but what is the minimum number of edges possible for $G$? (Hint: think about $\overline{G}$.) What if we want $\alpha(G) \leq r$, what is the minimum number of edges that $G$ can have then?
Let $T_{n,r}$ be the number of edges in the Turan graph on $n$ vertices and with $r$ parts. Turan’s theorem states that if $G$ is a simple graph on $n$ vertices and $\omega(G) \leq r$ then $m \leq T_{n,r}$.

Let $m$ be the minimum number of edges that a simple graph $H$ on $n$ vertices can have with $\alpha(H) \leq 2$ and let $G$ be an example of such a graph. Consider the graph $H = \overline{G}$. Since $\omega(H) = \omega(\overline{G}) = \alpha(G) \leq 2$, by Turan’s thm, $H$ has $\left(\begin{array}{c}n \\ 2\end{array}\right) - m \leq T_{n,2}$ edges. Thus $m \geq \left(\begin{array}{c}n \\ 2\end{array}\right) - T_{n,2}$.

Similarly if $\alpha(G) \leq r$ then $m \geq \left(\begin{array}{c}n \\ 2\end{array}\right) - T_{n,r}$.

4. (5.1.1) Let $G$ be the graph given by $V(G) = \{x, a, b, c, A, B, C\}$, $E(G) = \{xa, xb, ab, ac, bc, xA, xB, AB, AC, BC, cC\}$. What is $\alpha(G), \omega(G), \chi(G)$?

You must prove your answers. (How does $\chi$ compare with the two bounds $\chi \geq n/\alpha$ and $\chi \geq \omega$?)

$\alpha(G) \leq 2$. If the independent set contains $x$ then it can contain at most one point from the clique $c, C$. If the independent set does not contain $x$ then it can contain at most one from each of the cliques $a, b, c$ and $A, B, C$.

$\omega(G) = 3$. There are triangles (say $abc$) so $\omega(G) \geq 3$. One easily checks that for each point, the neighborhood of that point is triangle free, so $\omega(G) < 4$.

We know $\chi(G) \geq \omega(G) = 3$. Suppose $G$ is 3-colorable. Then $x$ is colored 1 without loss of generality and $a$ and $b$ are colored 2 and 3. This means $c$ is colored 1. But the same argument implies $A$ and $B$ are colored 2 and 3 and $C$ is colored 1, a contradiction. So $\chi(G) > 3$. However taking the previous bad coloring and recoloring $C$ to be color 4 we get a proper 4-coloring and so $\chi(G) = 4$.

The first bound is $\chi \geq n/\alpha = 7/2 = 3.5$ or $\chi \geq 4$, which is tight (as good as possible).

The second is $\chi \geq \omega$ or $\chi \geq 3$ which is not tight.