1. Starting with the zero flow, produce a maximum value flow in $N$ and a minimum value $s, t$ cut. Also record each flow-augmenting path you use.

The algorithm to find a flow augmenting path produces:

order placed on $L$ :  1  2  3  4  5  6  7  
vertex:  $s$ $a$ $b$ $c$ $d$ $g$ $t$  
label:  source $s$ $s$ $a$ $a$ $b$ $d$  

and a flow augmenting path $s, a, d, t$ of tolerance 3 (you can push an extra 3 units of flow along it)

The second round produces:

order placed on $L$ :  1  2  3  4  5  6  7  
vertex:  $s$ $b$ $a$ $g$ $c$ $d$ $t$  
label:  source $s$ $b$ $b$ $a$ $c$ $g$  

and a flow augmenting path $s, b, g, t$ of tolerance 2.

The third round produces:

order placed on $L$ :  1  2  3  4  5  6  7  
vertex:  $s$ $b$ $a$ $c$ $d$ $g$ $t$  
label:  source $s$ $b$ $a$ $a$ $c$ $d$  

and a flow-augmenting path $s, b, a, d, t$ of tolerance 1.

Currently we have $f(b, a) = 1$, $f(b, g) = f(g, t) = 2$, $f(s, b) = f(s, a) = 3$, $f(a, d) = f(d, t) = 6$ and $f = 0$ on every other edge and val($f$) = 6

The fourth round produces:

order placed on $L$ :  1  
vertex:  $s$  
label:  source  

Thus $S = \{s\}$, $T = \{a, b, c, d, g, t\}$ is an $s$-$t$ cut of capacity $c(S, T) = 6$.

Note that this equals the current flow value of 6. So we have produced a flow of maximum value.

2. (4.3.10) Let $G$ be a bipartite graph. Use network flows to prove that $\nu(G) = \tau(G)$.
Since the algorithm is beginning with integer flow and capacity values, we know that the max-flow algorithm will produce a flow of maximum value that takes on integer values. (What about the infinite capacities? You can make each infinite capacity a sufficiently large integer (in this case $|X| + 1$) and everything will be fine.) So our flow is integer valued.

Suppose two edges of $M$ meet in a vertex $y \in Y$. This would mean there are at least two units of flow entering $y$. But only the edge $(y, t)$ can carry this flow away and it has capacity 1, a contradiction. A similar contradiction is reached if two edges of $M$ meet in a vertex $x \in X$. Thus $M$ is a matching and $\text{val}(f) = |M| \leq \nu$.

Suppose $(S, T)$ is a cut of minimum capacity. Whatever this minimum capacity is, it is finite since $(\{s\}, V - s)$ is an $s$-$t$ cut of capacity $\leq |X| < \infty$. If there is an edge $e = xy$ of $G$ from $S \cap X$ to $T \cap Y$ then its capacity would contribute to $c(S, T)$. Since $e$ has infinite capacity this would contradict the fact that $(S, T)$ is a cut of minimum capacity.

Since there are no edges from $S \cap X$ to $T \cap Y$, the remaining vertices, $C = (T \cap X) \cup (S \cap Y)$ must cover all the edges, and so $\tau \leq |C|$. The capacity of the cut is the sum of the capacities of edges from $S$ to $T$. Since there are no edges from $S \cap X$ to $T \cap Y$ we are only summing the capacities of the edges from $s$ to $T \cap X$ and from $S \cap Y$ to $t$. Since each of these edges is of capacity 1 the total capacity of the cut is $|T \cup X| + |S \cap Y| = |C|$. Thus we have $\tau \leq |C| = \text{min cap} = \text{max flow} \leq \nu$. Since we always have $\nu \leq \tau$, this means $\nu = \tau$.

3. Let $P$ be the Petersen graph. What is $\kappa(P)$? What is $\kappa'(P)$? Justify your answers. (Don’t forget the theorem we mentioned in class relating $\kappa$, $\kappa'$, and $\delta$.)

We’ll show $\kappa(P) \geq 3$. We know by Whitney’s Theorem that $\kappa(G) \leq \kappa'(G) \leq \delta(G)$. So then $\kappa(P) = \kappa'(P) = \delta(P) = 3$.

To show that $\kappa(P) \geq 3$ we have to show that removing any two points leaves a connected graph. Recall the usual picture of the Petersen graph. Let $V_1 = \{a, b, c, d, e\}$, $V_2 = \{A, B, C, D, E\}$, $E_1 = \{ab, bc, cd, de, ae\}$, $E_2 = \{AB, BC, CD, DE, AE\}$. Then $V(P) = V_1 \cup V_2$, $E(P) = E_1 \cup E_2 \cup M$ where $M = \{aA, bC, cE, dB, eD\}$. In other words, $P$ consists of two $C_5$’s one on $V_1$ and the other on $V_2$ that are...
connected by a specific matching $M$. Select any two points $x, y$ of $P$. If one comes from $V_1$ and the other from $V_2$, then after their removal the $C_5$’s become $P_4$’s connected by at least one edge of $M$, hence $P - x - y$ remains connected. If $x, y$ come from $V_1$ then $P - x - y$ is a $C_5$ on $V_2$ and every other vertex is connected by an edge of $M$ to this $C_5$, so once again $P$ is connected.

4. Let $Q_k$ be the boolean cube. Example 4.1.3 shows that $\kappa(Q_k) = k$. Prove that between any pair of distinct vertices of $Q_k$ there are $k$ pairwise edge-disjoint paths.

Since $\kappa \leq \kappa' \leq \delta$ and $\kappa(Q_k) = \delta(Q_k) = k$ we have $\kappa'(Q_k) = k$. Thus if $x, y$ are a pair of distinct vertices, we have $\kappa'(x, y) \geq k$. By Menger’s theorem, $\lambda'(x, y) = \kappa'(x, y)$ and so $\lambda'(x, y) \geq k$. 