Test 1 solutions

1. Twenty children are playing at recess.

(a) How many ways can they form two teams of ten each to play
dodgeball? (Assume the teams are distinct.)
\[ \binom{20}{10} \] You have to choose 10 people out of 20 people to be members
of the first team. (The other 10 will be members of the second
team.)
Another way. You have to distribute 10 identical things to 20
distinct people, so that each person gets at most one. (Each person
who gets a thing is in the first team. Everyone who doesn’t is in
the second team.)

(b) How many ways can they form four teams? (Assume the teams
are distinct and each team has at least one member.)
\[ S(20, 4) \cdot 4! \] You have to assign 20 distinct people to 4 distinct
teams, and each team should get at least one person.

(c) They are now asked to line up into four different lines, one line
for each of four teachers. How many ways can this happen?
\[ (20 + 4 - 1)^{20} \] You have to assign 20 distinct people to 4 distinct
lines. The order in which people are assigned to a line counts.

2. You have ten gems, all different colors. How many different necklaces
of six gems can you make? (A necklace consists of gems attached to
evenly spaced points on a closed chain.)
\[ \binom{10}{6} \cdot \binom{12}{6} \] There are (10)\textsubscript{6} ways to assign
distinct gems to the points around
the necklace if the necklace is fixed in position. (View the necklace as
a regular hexagon with its 6 vertices labelled with distinct integers from
1 to 10.) Let’s say two fixed necklaces are equivalent if you can obtain
one from the other, by moving it around in space. Each fixed necklace
is then equivalent to 12 others. (You may rotate the necklace by any
multiple of 60 degrees and then you may subsequently flip it about
some fixed axis.)

3. (a) Write the inclusion exclusion formula for \(|A \cup B \cup C|\).

\[ |A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|. \]
(b) Suppose Mr. X joins an online dating service with a total enrollment of seven thousand members (including him). After a few computer searches of the complete membership (including him), he discovers that four thousand members have bad hair, four thousand have bad teeth, and four thousand have a bad attitude. Also, for each pair of these attributes two thousand members have both. Most disturbingly, one thousand have all three. How many have none of these properties?

0. Let \(A, B, C\) be the sets of members with bad hair, bad teeth, and a bad attitude, respectively. Then \(|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C| = 4000 + 4000 + 4000 - 2000 - 2000 - 2000 + 1000 = 7000\). So the number of members with none of these attributes is \(7000 - |A \cup B \cup C| = 0\). Mr. X learns (perhaps for the first time) that not even he is free of all of these properties.

**Bonus Problems:**

1. Prove \(S(k, n) = \sum_{j=0}^{n} (-1)^j \binom{n}{j} (n-j)^k\). (Hint: inclusion-exclusion.)

Incorrect as stated. \(S(k, n) = (1/n!) \sum_{j=0}^{n} (-1)^j \binom{n}{j} (n-j)^k\).

\(n!S(k, n)\) is the number of onto functions from \(X = \{1, 2, \ldots, k\}\) to \(Y = \{1, 2, \ldots, n\}\). For \(i = 1, 2, \ldots, n\), let \(A_i\) be the set of functions from \(X\) to \(Y\) such that \(i \not\in f(X)\). The set of functions from \(X\) to \(Y\) that are not onto is \(A_1 \cup A_2 \cup \cdots \cup A_n\). Thus \(n!S(k, n) = n^k - |A_1 \cup \cdots \cup A_n| = n^k - \sum_{j=1}^{n} (-1)^{j-1} \binom{n}{j} (n-j)^k = \sum_{j=0}^{n} (-1)^j \binom{n}{j} (n-j)^k\), by the inclusion-exclusion formula. (Note \(|A_i\cap A_{i_2} \cap \cdots \cap A_{i^j}| = (n-j)^k\).

2. Prove the inclusion-exclusion formula. (Hint: you can do it any way you like but here is an induction approach. \(|A_1 \cup \cdots \cup A_n| = |B \cup C| = |B| + |C| - |B \cap C|\) where \(B = A_1 \cup \cdots \cup A_{n-1}\) and \(C = A_n\) and use induction on \(|B|\) and \(|B \cap C|\).)

Let \([n] := \{1, 2, \ldots, n\}\). For \(S \subset [n]\), let \(A_S = \cap_{i \in S} A_i\). Then the inclusion-exclusion formula states \(|\cup_{i \in [n]} A_i| = \sum_{S \subset [n], S \neq \emptyset} (-1)^{|S|} |A_S|\).

For \(n = 2\), this is \(|A_1 \cup A_2| = |A_1| + |A_2| - |A_1 \cap A_2|\) which is easy to prove. Suppose now that the formula holds for \(n - 1\) and we wish to show it for \(n\). We follow the hint. \(|\cup_{i \in [n]} A_i| = |(\cup_{i \in [n-1]} A_i) \cup A_n| = |\cup_{i \in [n-1]} A_i| + |A_n| - |(\cup_{i \in [n-1]} A_i) \cap A_n|\) where the last equality comes
from the case \( n = 2 \). Now \( |\bigcup_{i \in \{n-1\}} A_i| = \sum_{T \subseteq \{n-1\}, T \neq \emptyset} (-1)^{|T|} |A_T| \) by the case \( n - 1 \). We can also use the case \( n - 1 \) to show \( |(\bigcup_{i \in \{n-1\}} A_i) \cap A_n| = |\bigcup_{i \in \{n-1\}} (A_i \cap A_n)| = \sum_{T \subseteq \{n-1\}, T \neq \emptyset} (-1)^{|T|} |A_T \cap A_n| \). Putting it all together we get

\[
|\bigcup_{i \in \{n\}} A_i| = \left( \sum_{T \subseteq \{n-1\}, T \neq \emptyset} (-1)^{|T|} |A_T| \right) + |A_n| - \sum_{T \subseteq \{n-1\}, T \neq \emptyset} (-1)^{|T|} |A_T \cap A_n| \\
= \sum_{S \subseteq \{n\}, S \neq \emptyset} (-1)^{|S|} |A_S|.
\]

3. How many ways can you distribute twenty golf balls to five golfers so that golfer one gets at least one, golfer two gets at least two, etc.?

\[
\begin{align*}
(20-(1+2+3+4+5)+(5-1)) &= (\frac{5+(5-1)}{5}) \\
&= (\frac{5+6}{5}).
\end{align*}
\]

Give the first golfer one ball, the second two balls, etc. Afterwards, you have 5 remaining identical golf balls to distribute to 5 distinct golfers.

4. Prove \( S(k,n) = S(k-1,n-1) + nS(k-1,n) \). (Assume \( k > n \geq 2 \)).

The partition may either have \( k \) in a part by itself or have \( k \) in a part with other elements. A partition of the first type can be created in \( S(k-1,n-1) \) ways. (Pick a partition \( P' \) of \( \{1,2,\ldots,k-1\} \) into \( n-1 \) parts and then add the part \( \{k\} \) to \( P' \) to get a partition \( P \) of \( \{1,2,\ldots,k\} \) into \( n \) parts, with \( \{k\} \) as a part.) A partition of the second type can be created in \( nS(k-1,n) \) ways. (Pick a partition \( P' \) of \( \{1,2,\ldots,k-1\} \) into \( n \) parts \( S(k-1,n) \) ways) and then add \( k \) to one of these \( n \) parts (times \( n \) additional ways) to get a partition \( P \) of \( \{1,2,\ldots,k\} \) into \( n \) parts in which \( k \) does not occur in a part by itself.) Thus \( S(k,n) = S(k-1,n-1) + nS(k-1,n) \).