HW 1: assigned 28 Aug 2002, due 4 Sept 2002, hand in during class on stapled loose leaf paper

1. (a) \( \binom{43}{2} = 903 \) edges. An edge can be viewed as a subset of 2 vertices out of 43.
(b) \( 2^{903} \) colorings. Each edge can be colored one of two colors.
(c) \( \binom{43}{5} \) \( K_5 \)'s. A \( K_5 \) can be viewed as a subset of 5 vertices out of 43.
(d) \( 2.064 \cdot 10^{260} \) years \( = 2^{903} \binom{43}{5} \) \( K_5 \)'s \( \frac{\sec}{10^{10} K_5} \cdot \frac{\text{year}}{3.1536 \times 10^7 \text{sec}} \).

2. There are only \( n \) possible remainders \( r \) when you divide a number \( a \) by \( n \), namely 0, 1, \ldots, \( n - 1 \). So out of \( n + 1 \) numbers two of them must have the same remainder, say \( a_i = bn + r \) and \( a_j = cn + r \), where \( i \neq j \).
So \( a_i - a_j = (b - c)n \) and \( n \) divides \( a_i - a_j \).

3. 7560. \( (x + 2y + 3)^7 = \sum_{i+j+k=7, i,j,k \geq 0} \binom{7}{i,j,k} x^i(2y)^j3^k \). The only term in this sum that contains the monomial \( x^3y^2 \) is the one where \( i = 3, j = 2, k = 2 \) or \( \binom{7}{3,2,2} x^3(2y)^23^2 \) = 7560 \( x^3y^2 \). So the coefficient of \( x^3y^2 \) is 7560.

4. If you pick \( k \) things from \( m + n \) things, you will end up picking \( i \) things from the first \( m \) things and \( j \) things from the last \( n \) things where \( i + j = k \). If you fix \( i, j \) then the number of ways to do this is \( \binom{m}{i} \binom{n}{j} \).
Thus \( \binom{m+n}{k} = \sum_{i+j=k, i,j \geq 0} \binom{m}{i} \binom{n}{j} \).

Another way to look at it. How many paths are there from (0, 0) to \( (k, m+n-k) \) using steps of the form (0, 1) or (1, 0) \( \binom{m+n}{k} \). Each path hits exactly one point in the set \( \{(i, m-i) : 0 \leq i \leq k\} \). How many ways can you get from (0, 0) to \( (i, m-i) \) \( \binom{i+(m-i)}{i} = \binom{m}{i} \). How many ways can you get from \( (i, m-i) \) to \( (k-i) \) \( \binom{k-i+(m-k+i)}{k-i} = \binom{n}{k-i} \).
Thus the number of paths that hit \( (i, m-i) \) is \( \binom{m}{i} \binom{n}{k-i} \). This means that \( \binom{m+n}{k} = \sum_{i=0}^{k} \binom{m}{i} \binom{n}{k-i} \).